Advances in Mathematics **157**, 138–176 (2001)
doi:10.1006/aima.2000.1955, available online at http://www.idealibrary.com on IDEAL®

Universality of Taylor Series as a Generic Property of Holomorphic Functions

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Received November 5, 1999; accepted September 14, 2000

1. INTRODUCTION

Let $\sum_{n=-\infty}^{+\infty} a_n e^{in\theta}$, $a_n \in \mathbb{C}$, be a formal trigonometric series; we denote [illegible]

property:

For every pair of Lebesgue measurable functions $h, g\colon [0, 2\pi] \to [-\infty, +\infty]$ there exists an increasing sequence $\lambda_n \in \{1, 2, 3, ...\}$ such that $Re\, S_{\lambda_n}(e^{i\theta}) \to h(\theta)$ and $Im\, S_{\lambda_n}(e^{i\theta}) \to g(\theta)$, as $n \to +\infty$, almost everywhere.

Such trigonometric series are called universal trigonometric series [17, 2].

In 1970 [14] Luh proved the existence of universal Taylor series with respect to summability matrices.

For $f_0(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ we write $S_\nu(z) = \sum_{k=0}^{\nu} a_k z^k$. Luh considers triangular matrices $A = (a_{nk})_{n,k=0}^{\infty}$, $a_{nk} \in \mathbb{C}$, $a_{nk} = 0$ for $k > n$ satisfying the two conditions

(i) $\lim_{n \to +\infty} a_{n\nu} = 0$

(ii) $\lim_{n \to +\infty} \sum_{\nu=0}^{n} a_{n\nu} = 1.$

The main result of Luh is the following:

There exists a power series $f_0(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ with radius of convergence 1, such that, for every bounded simply connected domain G, $G \cap \{z \in \mathbb{C} : |z| \leqslant 1\} = \varnothing$ and for every function $f\colon G \to \mathbb{C}$ holomorphic in G ($f \in H(G)$), there exists a strictly increasing sequence $n_k \in \{0, 1, 2, ...\}$ such that $\sigma_{n_k} \equiv \sum_{\nu=0}^{n_k} a_{n_k \nu} S_\nu(z)$ converges to f, as $k \to +\infty$, uniformly on compact subsets of G.

In 1971 Chui and Parnes, independently from Luh, proved the existence of a power series $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ with radius of convergence 1, such that, for every compact set K, $K \cap \{z \in \mathbb{C} : |z| \leqslant 1\} = \varnothing$ with K^c connected and for every function $h\colon K \to \mathbb{C}$ continuous on K and holomorphic in K^0, there

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exists a strictly increasing sequence $\lambda_n \in \{0, 1, 2, ...\}$ such that $S_{\lambda_n}(z) \to h(z)$, as $n \to +\infty$, uniformly on K. They also obtain that the sequence a_v, $v = 0, 1, 2, ...$ can be bounded [3].

Let Ω be an open subset of $\mathbb{C}$. We denote by $H(\Omega)$ the space of all holomorphic functions on Ω. It is well known that $H(\Omega)$ with the topology of uniform convergence on compacta is a complete metrizable space [12]. Now for $\zeta \in \Omega$ and $f \in H(\Omega)$ we denote by $\sum_{n=0}^{\infty} a_n(f, \zeta)(z-\zeta)^n$ the Taylor development of f with center ζ. Let also $S_n(f, \zeta)(z) = \sum_{k=0}^{n} a_k(f, \zeta)(z-\zeta)^k$, $n = 0, 1, 2, ...$ denote the partial sums. Luh in 1986 proved the following [13].

THEOREM 1.1. *Let $\Omega \subset \mathbb{C}$, $\Omega \neq C$ be an open set with simply connected components. Then there exists a function $f \in H(\Omega)$ and a strictly increasing sequence $\lambda_n \in \{0, 1, 2, ...\}$, such that, the following hold:*

(1) *For every $\zeta \in \Omega$ the sequence $S_{\lambda_n}(f, \zeta)(z)$ converges to $f(z)$, as $n \to +\infty$, uniformly on each compact subset of Ω.*

(2) *For every compact set K with $K \cap \bar{\Omega} = \varnothing$ and K^c connected and every function h: $K \to \mathbb{C}$ continuous on K and holomorphic in K^0, there exists a subsequence λ_{n_k}, $k = 1, 2, ...$, such that, for every $\zeta \in \Omega$ the sequence $S_{\lambda_{nk}}(f, \zeta)(z)$ converges to $h(z)$, as $k \to +\infty$, uniformly on K.*

Luh also proved that the set of functions $f \in H(\Omega)$ satisfying properties (1), (2) is dense in the space $H(\Omega)$ with the topology of uniform convergence on compacta.

We notice that in the previous results the approximation does not hold on the boundary of Ω, because K (or G) are disjoint from $\bar{\Omega}$. The convergence also in (2) is not assumed to be uniform with respect to the center ζ. The main tool in these results is the Mergelyans or Runge theorems and the method is constructive.

In 1996 in the case of the open unit disc and $\zeta = 0$ one of the authors included pieces of the boundary in the compact set K [18]. The reason for doing this was a question of Pichorides concerning approximation on the boundary by the partial sums in connection with rational functions (for details see [18]). The main tool is again Mergelyan's theorem [20], but the method is by Baire's theorem, whose use simplifies the proofs and yields that these universal properties are in fact generic (see also [7–10, 12, 16, 18]). The new class of universal Taylor series is compared with the class of Luh and Chui and Parnes in [16], where we see that they share some common properties, but they also have essential differences. For instance, one class meets the Disc Algebra, while the other is disjoint from the Nevanlinna class. Some properties of these universal Taylor series have been investigated in [11, 16, 18]. In [19] the notion of universal Taylor series is strengthened and extended in simply connected domains.

DEFINITION 1.2. Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, be a simply connected domain and $\zeta \in \Omega$. A holomorphic function $f \in H(\Omega)$ belongs to the class $U(\Omega, \zeta)$, iff for every compact set $K \subset \mathbb{C}$, such that $K \cap \Omega = \emptyset$ and K^c is connected and every function $h: K \to \mathbb{C}$ continuous on K and holomorphic in K^0, there exists a sequence $\lambda_n \in \{0, 1, 2, ...\}$ such that $S_{\lambda_n}(f, \zeta)(z)$ converges to $h(z)$, as $n \to +\infty$, uniformly on K.

DEFINITION 1.3. Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, be a simply connected domain. A holomorphic function $f \in H(\Omega)$ belongs to the class $U(\Omega)$, iff for every compact set $K \subset \mathbb{C}$, such that $K \cap \Omega = \emptyset$ and K^c is connected and for every function $h: K \to \mathbb{C}$ continuous on K and holomorphic in K^0, there exists a sequence $\lambda_n \in \{0, 1, 2, ...\}$, such that for every compact set $L \subset \Omega$ we have

$$\sup_{\zeta \in L} \sup_{z \in K} |S_{\lambda_n}(f, \zeta)(z) - h(z)| \to 0, \text{ as } n \to +\infty.$$

We notice that in the previous definitions it is equivalent to require that the sequence λ_n, $n = 1, 2, ...$ is strictly increasing. If Ω is not simply connected, then we prove that $U(\Omega) = \emptyset$ (Section 3). This justifies the fact that one has to assume that Ω is simply connected.

In [19] we proved that the classes $U(\Omega, \zeta)$ and $U(\Omega)$ are G_δ-dense in $H(\Omega)$. In particular they are non-void. In the present paper we generalize this result when $S_{\lambda_n}(f, \zeta)(z)$ is replaced by a matrix summation method. The matrices we consider vary analytically with z. Baire's theorem allows us to achieve simultaneously any denumerable set of generic properties. Thus, if we want to prove the existence of a holomorphic function satisfying various properties, it is enough to prove that each one of these properties is generic. This is not very complicated, in the cases we have treated. We use Baire's and Mergelyan's theorems and a few other elements of complex analysis. Furthermore the repetition simultaneously of the proofs of several generic properties leeds to stronger results, as the following theorem proved in Section 4 shows.

THEOREM 1.4. *Let Ω be a simply connected domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ and S an infinite subset of $\{0, 1, 2, ...\}$. Then there exists a holomorphic function $f \in H(\Omega)$ such that the following hold:*

For every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \emptyset$, K^c connected, and every function $\phi: K \to \mathbb{C}$ continuous on K and holomorphic in K^0, there exists a strictly increasing sequence $\lambda_n \in S$, $n = 1, 2, ...$, such that for every compact sets $L, \tilde{L} \subset \Omega$ and $J \subset (-1, +\infty)$ we have

(1) $\sup_{\zeta \in L} \sup_{w \in \tilde{L}} |S_{\lambda_n}(f, \zeta)(w) - f(w)| \to 0$, *as* $n \to +\infty$ *and*

(2) $\sup_{a \in J} \sup_{\zeta \in L} \sup_{z \in K} |\sigma^a_{\lambda_n}(f, \zeta)(z) - \phi(z)| \to 0$, *as* $n \to +\infty$,

where $\sigma_n^a(f, \zeta)(z)$ *is the* (C, a) *mean of the Taylor development of* f *with center* ζ.

Furthermore, the set of functions f *with this property is* G_δ*-dense in* $H(\Omega)$ *endowed with the topology of uniform convergence on compacta.*

The previous results are contained in Sections 3 and 4. In Section 5 we establish some properties of universal Taylor series, as for example:

(1) For every $f \in H(\Omega)$, there exists $u_1, u_2 \in U(\Omega)$ such that $f = u_1 - u_2$.

(2) Every $f \in U(D, 0)$, where D is the open unit disk, is a universal trigonometric series in the sense of D. Menchoff. This result generalizes to the class $U(\Omega, \zeta)$ when $(\partial\Omega)^c$ has a locally finite number of components.

(3) If $f \in U(\Omega, \zeta)$, then f is not a rational function.

(4) If $f \in U(\Omega, \zeta)$, then its Taylor development $\sum_{n=0}^{\infty} a_n(f, \zeta)(z - \zeta)^n$ is not (C, k) summable at any $z \in \partial\Omega$ and for any $k = 1, 2, \ldots$.

(5) If $f \in U(\Omega, \zeta)$, then f does not extend continuously on $\bar{\Omega}$.

(6) If $f \in U(\Omega)$, then f does not extend holomorphicaly accross the boundary of Ω.

The last property relates to a conjecture of J.-P. Kahane [10], which states that any element of $U(D, 0)$ has as natural boundary the unit circle.

In Section 6 we transfer the previous results to the derivatives and antiderivatives of holomorphic functions.

In Section 7 we consider the case of a simply connected open set with several components.

Section 8 contains the answer to the above mentioned conjecture of Kahane, which is in the affirmative. The proof is based on a recent result of Gehlen *et al.* [5].

In Section 9 we prove that $U(\Omega, \zeta) = U(\Omega)$ provided that Ω is a domain in $\mathbb{C}$ contained is some half–plane.

First, in Section 2 we prove some preliminary results that we use in our proofs.

2. PRELIMINARIES

In this section we establish some preliminary results that we will use in Sections 3, 4, 5, 6, and 7.

Let Ω be an open subset of $\mathbb{C}$. The fact that the components of Ω are simply connected is equivalent to saying that $(C \cup \{\infty\}) \backslash \Omega$ is connected. This equivalence, in the case where Ω is a domain, has been proved, for example, in [1]. Essentially the same argument applies to the general case.

We also refer to open sets with simply connected components, as simply connected open sets.

Let Ω be an open subset of $\mathbb{C}$ with components $\Omega_1, \Omega_2, \ldots$. We say that the number of components of Ω is locally finite, if, for every compact set $L \subset \mathbb{C}$, the set $\{n: L \cap \Omega_n \neq \emptyset\}$ is finite.

LEMMA 2.1. *Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ be an open set. We suppose that the number of components of Ω is locally finite. Then there exists a sequence $K_m \subset \mathbb{C}$, $m = 1, 2, \ldots$ of compact sets, $K_m \cap \Omega = \emptyset$, with K_m^c connected, such that for every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \emptyset$ with K^c connected, there exists $m = 1, 2, \ldots$ with $K \subset K_m$.*

Proof. Let $\Omega_1, \Omega_2, \ldots$ be the component of Ω. Let K be compact $K \subset \mathbb{C}$, $K \cap \Omega = \emptyset$ with K^c connected. Let $N \in \{1, 2, \ldots\}$ be such that $K \subset \{z \in \mathbb{C} : |z| \leqslant N\}$. The closed disk $\{z \in \mathbb{C} : |z| \leqslant N\}$ intersects a finite number of $\Omega_1, \Omega_2, \ldots$. Let $m_0 \in \{1, 2, \ldots\}$ be such that, $\{z \in \mathbb{C} : |z| \leqslant N\} \cap \Omega_j = \emptyset$ for all $j = m_0 + 1, m_0 + 2, \ldots$. For every $j = 1, \ldots, m_0$ we consider a polygonal arc $\Gamma_j \subset K^c$ whose vertices have rational coordinates, starting in Ω_j and ending at $N+1$. We set $\Gamma = \bigcup_{j=1}^{m_0} \Gamma_j$ and let $s \in \{1, 2, \ldots\}$ such that $dist(\Gamma, K) > \frac{1}{s}$. Then K is contained in $L(N, m_0, \Gamma, s) = \{z \in \Omega^c : |z| \leqslant N,\ dist(z, \Gamma) \geqslant \frac{1}{s}\}$.

The set of all possible $L(N, m, \Gamma, s)$ is obviously denumerable; thus, it can be enumerated and gives the sequence K_m, $m = 1, 2, \ldots$. It remains to prove that each $[L(N, m_0, \Gamma, s)]^c$ is connected. Indeed $[L(N, m_0, \Gamma, s)]^c = \Omega \cup \{z \in \mathbb{C} : |z| > N\} \cup \{z \in \mathbb{C} : \mathrm{dist}(z, \Gamma) < \frac{1}{s}\} = \{z \in \mathbb{C} : |z| > N\} \cup \{z \in \mathbb{C} : \mathrm{dist}(z, \Gamma) < \frac{1}{s}\} \cup \Omega_1 \cup \cdots \cup \Omega_{m_0}$.

Each one of the sets in the last expression is connected and the set $\{z \in \mathbb{C} : \mathrm{dist}(z, \Gamma) < \frac{1}{s}\}$ intersects all other sets. Thus $[L(N, m_0, \Gamma, s)]^c$ is connected. This completes the proof. ∎

If the number of components of Ω is not locally finite, the previous result in general fails. This can be seen by the following.

COUNTEREXAMPLE. For $n = 1, 2, \ldots$ we consider the open disk

$$\Omega_n = \left\{ z \in \mathbb{C} : \left| z - \frac{1}{2^n} \right| < \frac{1}{2^{n+2}} \right\} \text{ and let } \Omega = \bigcup_{n=1}^{\infty} \Omega_n.$$

Suppose that there exists a sequence K_m, $m = 1, 2, \ldots$ of compact sets, $K_m \cap \Omega = \emptyset$ with K_m^c connected, such that every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \emptyset$, K^c connected, is contained in some K_m. We shall obtain a contradiction. For $n = 1, 2, \ldots$ we set $\Gamma_n(1) = \{z \in \partial\Omega_n : Im\, z \geqslant 0\}$ and $\Gamma_n(-1) = \{z \in \partial\Omega_n : Im\, z \leqslant 0\}$. For every $\varepsilon = (\varepsilon_n)_{n=1}^{\infty} \in \{-1, 1\}^{\omega}$, we consider the compact set $L(\varepsilon) = \{0\} \cup \bigcup_{n=1}^{\infty} \Gamma_n(\varepsilon_n)$. Obviously $L(\varepsilon) \cap \Omega = \emptyset$

and $L(\varepsilon)^c$ is connected. Thus, there exists $m(\varepsilon) \in \{1, 2, ...\}$, such that $L(\varepsilon) \subset K_{m(\varepsilon)}$.

The correspondence $\{-1, 1\}^\omega \ni \varepsilon \to m(\varepsilon) \in \{1, 2, ...\}$ is one to one. Indeed let $\varepsilon, \varepsilon' \in \{-1, 1\}^\omega$, $\varepsilon \neq \varepsilon'$ be such that $m(\varepsilon) = m(\varepsilon')$. Then there exists n_0, such that $\varepsilon_{n_0} \neq \varepsilon'_{n_0}$; we also have

$$\Gamma_{n_0}(\varepsilon_{n_0}) \subset L(\varepsilon) \subset K_{m(\varepsilon)} \qquad \text{and} \qquad \Gamma_{n_0}(\varepsilon'_{n_0}) \subset L(\varepsilon') \subset K_{m(\varepsilon')} = K_{m(\varepsilon)}.$$

Thus

$$\left\{ z \in \mathbb{C} : \left| z - \frac{1}{2^{n_0}} \right| = \frac{1}{2^{n_0+2}} \right\} = \Gamma_{n_0}(\varepsilon_{n_0}) \cup \Gamma_{n_0}(\varepsilon'_{n_0}) \subset K_{m(\varepsilon)}.$$

It follows that $K^c_{m(\varepsilon)}$ has a bounded component (the disk Ω_{n_0}). This gives a contradiction and proves that the correspondence $\{-1, 1\}^\omega \ni \varepsilon \to m(\varepsilon) \in \{1, 2, ...\}$ is one to one. However, this is impossible, because $\{-1, 1\}^\omega$ is non denumerable. The proof is completed. ∎

We also need the following.

LEMMA 2.2. *Let $F \subset \mathbb{C}$ be a closed set, such that F^c has locally finite number of components. Then there exists an increasing sequence of compact sets $K_n \subset F$, $n = 1, 2, ...$, with K_n^c connected, such that $\bigcup_{n=1}^\infty K_n = F$.*

Proof. We assume that F^c has an infinite number of components Ω_n, $n = 1, 2, ...$. The case of a finite number of components is simpler and is omitted.

From each component Ω_n, $n = 1, 2, ...$, we chose a point $z_n \in \Omega_n$. Since the number of components of F^c is locally finite we have $\lim_n |z_n| = +\infty$.

For every $m = 1, 2, ...$ we consider an open connected set O^m as follows. For $n = 1, 2, ...$ we consider the open annulus $\mathscr{D}^m_n = \{z : |z_n| < |z| < |z_n| + \frac{1}{m}\}$ and we set $\mathscr{D}^m = \bigcup_{n=1}^\infty \mathscr{D}^m_n$. Let also $A^m = \{z : 0 < Re\, z < \frac{1}{m}\}$. Finally we set $O^m = A^m \cup \mathscr{D}^m$. Obviously O^m is open connected and $O^{m+1} \subset O^m$. We also have $\bigcap_{m=1}^\infty O^m = \varnothing$.

Indeed, let $z \in \bigcap_{m=1}^\infty O^m$; then, for every $m = 1, 2, ...,$ we have $0 < Re\, z < \frac{1}{m}$ or there exists $n(m)$, such that $|z_{n(m)}| < |z| < |z_{n(m)}| + \frac{1}{m}$.

If for infinetily many m we had $0 < Re\, z < \frac{1}{m}$, then we conclude $Re\, z = 0$, which contradicts the inequality $0 < Re\, z$. Thus, for some m_0 and for every $m \geqslant m_0$, there exists $n(m)$ such that $|z_{n(m)}| < |z| < |z_{n(m)}| + \frac{1}{m}$. since $\lim_n |z_n| = +\infty$, we conclude that the sequence $n(m)$, $m \geqslant m_0$ is bounded. Therefore, there exists n_0 such that $n(m) = n_0$ for infinitely many m. Passing to the limit, as $m \to +\infty$, we take $|z| = |z_{n_0}|$, which contradicts the inequality $|z_{n_0}| = |z_{n(m)}| < |z|$. Therefore $\bigcap_{m=1}^\infty O^m = \varnothing$.

We set $K_m = \{z \in \mathbb{C} : |z| \leqslant m\} \cap F \cap (O^m)^c$, $m = 1, 2, \ldots$. Obviously K_m is compact, $K_m \subset K_{m+1}$, and $\bigcup_{m=1}^{\infty} K_m = F$. It remains to be seen that K_m^c is connected.

Indeed we have $K_m^c = \{z \in \mathbb{C} : |z| > m\} \cup O^m \cup (\bigcup_{n=1}^{\infty} \Omega_n)$. The sets $\{z \in \mathbb{C} : |z| > m\}$, O^m, and Ω_n, $n = 1, 2, \ldots$ are all connected and O^m intersects each Ω_n, $n = 1, 2, \ldots$, and also $\{z : |z| > m\}$. Thus, K_m^c is connected. This completes the proof. ∎

Katsoprinakis used Rogosinski's formula [21, Vol. I, p. 114] and proved that, if $f(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$ belongs to the class $U(D, 0)$, where D is the open unit disk, then for every z_0, $|z_0| = 1$, the series $\sum_{\nu=0}^{\infty} c_\nu z_0^\nu$ is not $(C, 1)$ summable (see [18]). Kahane suggested to obtain extensions of Rogosinski's formula and prove that $\sum_{\nu=0}^{\infty} c_\nu z_0^\nu$ is not (C, k) summable, for any $k = 1, 2, \ldots$. This has been realized in [11]. In the present paper we extend this result in the more general class $U(\Omega, \zeta)$, where Ω is any simply connected domain in $\mathbb{C}$, $\Omega \neq \mathbb{C}$. Towards this end we use a formula similar to the formula established in [11]. For the purpose of completeness we include the proof of this formula.

Lemma 2.3. *Let a_n, $n = 0, 1, 2\ldots$ be a sequence of complex numbers and for $n = 0, 1, 2, \ldots$ we set $A_n = a_0 + a_1 + \cdots + a_n$. Then, for every $n = 1, 2, \ldots$ and $z \in \mathbb{C}$ we have*

$$\sum_{\nu=0}^{n} a_\nu z^\nu = \sum_{\nu=0}^{n-1} A_\nu (1 - z)\, z^\nu + A_n z^n.$$

The proof of this lemma is standard and is omitted.

If C_n, $n = 0, 1, 2, \ldots$, is a sequence of complex numbers we define the sequences $S_n^0 \equiv S_n, S_n^1, \ldots$ as follows:

$$S_n^0 = c_0 + \cdots + c_n,\ S_n^1 = S_0 + \cdots + S_n,\ \ldots,\ S_n^{k+1} = S_0^k + \cdots + S_n^k.$$

We also define $A_n^0 = 1$, $A_n^1 = A_0^0 + \cdots + A_n^0, \ldots, A_n^{k+1} = A_0^k + \cdots + A_n^k$. It follows that $A_n^j = \binom{n+j}{j}$.

The series $\sum_{\nu=0}^{\infty} c_\nu$ is (C, k) summable to $s \in \mathbb{C}$ iff $\lim_{n \to +\infty} (S_n^k / A_n^k) = s$.

Lemma 2.4. *With the above notations, for every $k = 0, 1, 2, \ldots$, $n \geqslant k + 1$ and $z \in \mathbb{C}$ we have*

$$\sum_{\nu=0}^{n} c_\nu z^\nu = \sum_{\nu=0}^{n-(k+1)} S_\nu^k (1 - z)^{k+1} z^\nu + \sum_{j=0}^{k} S_{n-j}^j\, z^{n-j} (1 - z)^j. \tag{2.1}$$

Proof. According to Lemma 2.3 we have

$$\sum_{v=0}^{n} c_v z^v = \sum_{v=0}^{n-1} S_v^0 (1-z)\, z^v + S_n^0\, z^n. \tag{I_0}$$

We also have $\sum_{v=0}^{n-1} S_v^0\, z^v = \sum_{v=0}^{n-2} S_v^1 (1-z)\, z^v + S_{n-1}^1\, z^{n-1}$. Multiplying this last equality by $(1-z)$ we get

$$\sum_{v=0}^{n-1} S_v^0 (1-z)\, z^v = \sum_{v=0}^{n-2} S_v^1 (1-z)^2\, z^v + S_{n-1}^1 (1-z)\, z^{n-1}.$$

More generally, for $j = 1, ..., k$ we have

$$\sum_{v=0}^{n-j} S_v^{j-1}\, z^v = \sum_{v=0}^{n-j-1} S_v^j (1-z)\, z^v + S_{n-j}^j\, z^{n-j}.$$

Multiplying this last equality by $(1-z)^j$ we get

$$\sum_{v=0}^{n-j} S_v^{j-1} (1-z)^j\, z^v = \sum_{v=0}^{n-j-1} S_v^j (1-z)^{j+1}\, z^v + S_{n-j}^j (1-z)^j\, z^{n-j}. \tag{I_j}$$

Adding (I_0), (I_1), ..., (I_k) we take

$$\sum_{v=0}^{n} c_n z^n = \sum_{v=0}^{n-(k+1)} S_v^k (1-z)^{k+1}\, z^v + \sum_{j=0}^{k} S_{n-j}^j (1-z)^j\, z^{n-j}.$$

This proves the lemma. ▮

Lemma 2.4 in the case $c_0 = 1$, $c_1 = c_2 = \cdots = 0$ gives

$$1 = \sum_{v=0}^{n-(k+1)} A_v^k (1-z)^{k+1}\, z^v + \sum_{j=0}^{k} A_{n-j}^j\, z^{n-j} (1-z)^j. \tag{2.2}$$

Suppose that $\sum_{v=0}^{\infty} c_v$ is (C, k) summable to $s \in \mathbb{C}$; then $\lim_{n\to\infty} (S_n^k / A_n^k) = s$. We set $S_n(z) = \sum_{v=0}^{n} c_v z^v$. We multiply (2.2) by s and we subtract from (2.1). Setting $\varepsilon_n = (S_n^k / A_n^k) - s \to 0$ (as $n \to +\infty$) we get

$$\begin{aligned}(S_n(z) - s) - (S_n - s)\, z^n = \sum_{v=0}^{n-(k+1)} \varepsilon_v A_v^k (1-z)^{k+1}\, z^v + \sum_{j=1}^{k-1} (S_{n-j}^j - s A_{n-j}^j) \\ \cdot (1-z)^j\, z^{n-j} + \varepsilon_{n-k} A_{n-k}^k (1-z)^k\, z^{n-k}.\end{aligned} \tag{2.3}$$

Let $\mathscr{D}$ be an infinite subset of $\{0, 1, 2, ...\}$. For $n \in \mathscr{D}$ we set $z = z_n$, and we assume that $|n(1 - z_n)|$, $n \in \mathscr{D}$ is bounded by $C < +\infty$. Then we have

$$|\varepsilon_{n-k} A_{n-k}^k (1 - z_n)^k z_n^{n-k}| \leqslant |\varepsilon_{n-k}| \binom{n}{k} \frac{C^k}{n^k} \left(1 + \frac{C}{n}\right)^{n-k} \leqslant C' |\varepsilon_{n-k}| \to 0,$$

as $n \to +\infty$, and

$$\left| \sum_{\nu=0}^{n-(k+1)} \varepsilon_\nu A_\nu^k (1 - z_n)^{k+1} z_n^\nu \right| \leqslant \sum_{\nu=0}^{n-(k+1)} |\varepsilon_\nu| \binom{\nu + k}{k} \frac{C^{k+1}}{n^{k+1}} \left(1 + \frac{c}{n}\right)^\nu$$
$$\leqslant \frac{C'}{n} \sum_{\nu=0}^{n-(k+1)} |\varepsilon_\nu| \to 0,$$

as $n \to +\infty$.

Therefore, we have proved the following.

PROPOSITION 2.5. Suppose that the series $\sum_{\nu=0}^\infty c_\nu$ is (C, k) summable to $s \in \mathbb{C}$. Let $\mathscr{D}$ be an infinite subset of $\{0, 1, 2, ...\}$ and for every $n \in \mathscr{D}$, let z_n be a complex number, such that $n(1 - z_n)$, $n \in \mathscr{D}$, is bounded. Then for every $n \in \mathscr{D}$ we have

$$S_n(z_n) - s - (S_n - s) z_n^n = \sum_{j=1}^{k-1} (S_{n-j}^j - sA_{n-j}^j)(1 - z_n)^j z_n^{n-j} + o(1),$$

as $n \to \infty$ $(n \in \mathscr{D})$.

LEMMA 2.6. *Let $a_1, ..., a_k$ be distinct complex numbers. Then for every $\tau = 0, 1, ..., k-2$ we have*

$$\sum_{\ell=1}^k \frac{a_\ell^\tau}{\prod_{r=1, r \neq \ell}^k (a_\ell - a_r)} = 0.$$

Proof. We know that there exists a unique polynomial P with $\deg P \leqslant k - 1$ satisfying $P(a_\ell) = a_\ell^\tau$ for all $\ell = 1, ..., k$. Obviously $P(z) \equiv z^\tau$. On the other hand, by the Lagrange interpolation formula, we have

$$z^\tau \equiv P(z) \equiv \sum_{\ell=1}^k \frac{a_\ell^\tau}{\prod_{r=1, r \neq \ell}^k (a_\ell - a_r)} \cdot \prod_{\substack{r=1 \\ r \neq \ell}}^k (z - a_r).$$

Equating the coefficients of z^{k-1} we obtain the result. ∎

LEMMA 2.7. *Let $\sum_{v=0}^{\infty} C_v$ be (C, k) summable to $s \in \mathbb{C}$. Let $\mathscr{D}$ be an infinite subset of $\{0, 1, 2, ...\}$. For every $n \in \mathscr{D}$ and $\ell = 1, ..., k$, let $z_{\ell, n}$ be a complex number, such that $\lim_{n \to +\infty, n \in \mathscr{D}} [n(1 - z_{\ell, n})] = \zeta_\ell \neq 0$. We assume that $\zeta_1, ..., \zeta_k$ are distinct.*

Then for every $n \in \mathscr{D}$ and $\ell = 1, ..., k$, there exists a complex number $\lambda_{\ell, n}$ such that

and (i) $$\lim_{\substack{n \to +\infty \\ n \in \mathscr{D}}} \sum_{\ell=1}^{k} [z_{\ell, n}^{-n}(S_n(z_{\ell, n}) - s) - (S_n - s)]\, \lambda_{\ell, n} = 0$$

(ii) $$\lim_{\substack{n \to +\infty \\ n \in \mathscr{D}}} \lambda_{\ell, n} = \frac{1}{\zeta_\ell \prod_{r=1, r \neq \ell}^{k} (\zeta_\ell - \zeta_r)}.$$

Proof. We set

$$\lambda_{\ell, n} = \frac{[n(z_{\ell, n}^{-1} - 1)]^{-1}}{\prod_{r=1, r \neq \ell}^{k} [1/n(z_{\ell, n}^{-1} - z_{r, n}^{-1})]}.$$

We easily obtain (ii).

We obtain (i) using Theorem 2.5 and the fact that $\sum_{\ell=1}^{k} \lambda_{\ell, n}(1 - z_{\ell, n})^j z_{\ell, n}^{-j} = 0$ according to Lemma 2.6. ∎

LEMMA 2.8. *Let $\Omega \subset \mathbb{C}$ be an open simply connected set, such that $1 \in \partial\Omega$. Then there exists a compact set $K \subset \mathbb{C}$, $K \cap \Omega = \varnothing$ with K^c connected, such that the following holds:*

For every infinite set $E \subset \{1, 2, ...\}$, for every $k = 1, 2, ...$ and arbitrary $\ell_1, ..., \ell_k \in \{1, 2, ...\}$, there exists an infinite set $\mathscr{D} \subset E$ and complex numbers $z_{j, n} \in K$, $n \in \mathscr{D}$, $j = 1, ..., k$, such that

$$\lim_{\substack{n \to +\infty \\ n \in \mathscr{D}}} [n(1 - z_{j, n})] = \zeta_j \quad \textit{and} \quad |\zeta_j| = \frac{1}{\ell_j} \quad \textit{for all} \quad j = 1, ..., k.$$

Proof. $(C \cup \{\infty\}) \setminus \Omega$ is connected and contains 1 and ∞. Therefore, for every $n = 1, 2, ...$ there exists $w_n \in \Omega^c$ such that $|1 - w_n| = \frac{1}{n}$. We set $K = \{1\} \cup \{w_n : n = 1, 2, ...\}$. Obviously K is compact, $K \cap \Omega = \varnothing$ and K^c is connected. Let $\ell_1, ..., \ell_k$ be natural numbers. For $j = 1, ..., k$ and $n = 1, 2, ...$ we set $z_{j, n} = w_{\ell_j n}$; therefore $|n(1 - z_{j, n})| = \frac{1}{\ell_j}$.

The sequences $n(1 - z_{j, n})$, $n \in E$, $j = 1, ..., k$ are bounded. Therefore, there exists an infinite set $\mathscr{D} \subset E$, such that $\lim_{n \to +\infty, n \to +\infty\, n \in \mathscr{D}} [n(1 - z_{j, n})] = \zeta_j$, $j = 1, ..., k$. Obviously $|\zeta_j| = \frac{1}{\ell_j}$. This completes the proof. ∎

Next we develop a method allowing us to transfer generic properties of holomorphic functions to their derivatives or antiderivatives.

Let $\Omega \subset \mathbb{C}$ be a simply connected open set, with components Ω_j, $j \in J$ where the set J is finite or infinite denumerable. We choose a point z_j from each component Ω_j, $j \in J$. We fix this choice $z_j \in \Omega_j$, $j \in J$. $H(\Omega)$ denotes the space of holomorphic functions in Ω under the topology of uniform convergence on compacta (which is a complete metrizable space). We also denote as $H^o(\Omega)$ the closed subspace of $H(\Omega)$ containing the functions $f \in H(\Omega)$ satisfying $f(z_j) = 0$ for all $j \in J$. We consider the map π: $H(\Omega) \to H^o(\Omega)$ where $\pi(f)$ satisfies $\pi(f)(z) = f(z) - f(z_j)$ for every $z \in \Omega_j$ and all $j \in J$. Obviously $\pi|_{H^o(\Omega)}$ is the identity map.

LEMMA 2.9. *Under the above assumptions and notations the map* π: $H(\Omega) \to H^o(\Omega)$ *is linear, onto, continuous and open.*

Proof. Since π is continuous and surjective, the result follows from the open mapping theorem. ▮

We consider now the map Φ: $H^o(\Omega) \to H(\Omega)$ defined by $\Phi(f) = f'$ for all $f \in H^o(\Omega)$. Obviously Φ is linear continuous and one to one. Since Ω has simply connected components, the map Φ is onto (therefore bijective). The inverse map Φ^{-1}: $H(\Omega) \to H^o(\Omega)$ is defined by $[\Phi^{-1}(f)](z) = \int_{z_j}^{z} f(\zeta)\, d\zeta$, $z \in \Omega_j$ where the integral is independent of the path of integration in Ω_j, because Ω_j is simply connected.

LEMMA 2.10. *Under the above assumptions and notations the map* Φ: $H^o(\Omega) \to H(\Omega)$ *is a homeomorphism.*

Proof. Essentially we have to prove that the map Φ^{-1}: $H(\Omega) \to H^o(\Omega)$ is continuous. This follows also from the open mapping theorem. ▮

We consider now the map T: $H(\Omega) \to H(\Omega)$ given by $T(f) = f'$. Then $T = \Phi \circ \pi$ and T is linear, onto, continuous and open.

LEMMA 2.11. *Let A be a G_δ-dense set in $H(\Omega)$. Then for every* $n \in \{1, 2, ...\}$ *the set $T^{-n}(A)$ is G_δ-dense in $H(\Omega)$*

Proof. Let $A = \bigcap_{m=1}^{\infty} O_m$, where O_m is open dense in $H(\Omega)$. Then $T^{-1}(A) = \bigcap_{m=1}^{\infty} T^{-1}(O_m)$.

Since T is continuous, it follows that $T^{-1}(O_m)$ is open in $H(\Omega)$. Thus, $T^{-1}(A)$ is G_δ in $H(\Omega)$. We also have $T = \Phi \circ \pi$ and $T^{-1}(O_m) = \pi^{-1}(\Phi^{-1}(O_m))$. Since Φ is a homeomorphism, $\Phi^{-1}(O_m)$ is dense in $H^o(\Omega)$. Let $f \in H(\Omega)$; then $\pi(f) \in H^o(\Omega)$ is given by $\pi(f)(z) = f(z) - f(z_j)$ for $z \in \Omega_j$. Let h_k, $k = 1, 2, ...,$ $h_k \in \Phi^{-1}(O_m)$, be such that $h_k \to \pi(f)$, as $k \to +\infty$. We also define $g_k \in H(\Omega)$ by $g_k(z) = h_k(z) + f(z_j)$ for $z \in \Omega_j$. Then $g_k \in \pi^{-1}(h_k) \subset \pi^{-1}(\Phi^{-1}(O_m)) = T^{-1}(O_m)$ and $g_k \to f$, as $k \to +\infty$.

This shows that $T^{-1}(O_m)$ is also dense in $H(\Omega)$. Thus $T^{-1}(A)$ is G_δ-dense in $H(\Omega)$.

Inductively we find that $T^{-n}(A)$ is G_δ-dense in $H(\Omega)$, for all $n = 1, 2, \ldots$. ∎

We consider the map $\Psi: H(\Omega) \to H(\Omega)$ defined by $\Psi(f) = \Phi^{-1}(f)$, $f \in H(\Omega)$.

LEMMA 2.12. *Let $n \in \{1, 2, \ldots\}$ and let $A \subset H(\Omega)$ be G_δ-dense in $H(\Omega)$. We also assume that if $f \in A$ and $h \in H(\Omega)$ is such that $h|_{\Omega_j}$ is a polynomial of degree at most $n-1$, for all $j \in J$, then $f + h \in A$. Under these assumptions the set $\Psi^{-n}(A)$ is G_δ-dense in $H(\Omega)$.*

Proof. Let first assume $n = 1$. Our assumption implies that $A \cap H^o(\Omega) = \pi(A)$. We also have $\Psi^{-1}(A) = \{f \in H(\Omega) : \Psi(f) \in A\} = \{f \in H(\Omega) : \Phi^{-1}(f) \in A\} = \{f \in H(\Omega) : \Phi^{-1}(f) \in A \cap H^o(\Omega)\} = \{g' \in H(\Omega) : g \in A \cap H^o(\Omega)\}$. Since $A \cap H^o(\Omega) = \pi(A)$, it follows that $\Psi^{-1}(A) = \{\Phi(g): g \in \pi(A)\} = \Phi(\pi(A)) = T(A)$.

The map $T: H(\Omega) \to H(\Omega)$ is continuous and onto. Since A is dense in $H(\Omega)$ and T is continuous onto, it follows that $T(A)$ is also dense. Thus, $\Psi^{-1}(A)$ is dense in $H(\Omega)$.

Let $A = \bigcap_{m=1}^{\infty} O_m$, where O_m are open in $H(\Omega)$. Then $\pi(A) = A \cap H^o(\Omega) = \bigcap_{m=1}^{\infty} [O_m \cap H^o(\Omega)]$ is G_δ in $H^o(\Omega)$. Since $\Phi: H^o(\Omega) \to H(\Omega)$ is a homeomorphism, it follows that $\Phi(\pi(A))$ is G_δ in $H(\Omega)$. Therefore, $\Psi^{-1}(A) = \Phi(\pi(A))$ is G_δ-dense in $H(\Omega)$.

This completes the proof in the case $n = 1$.

The general case follows by induction. Let $A \subset H(\Omega)$ be G_δ dense. We also assume that if $f \in A$ and $h \in H(\Omega)$ are such that $h|_{\Omega_j}$ is a polynomial of degree at most $n-1$, then $f + h \in A$.

We consider the set $\Psi^{-1}(A)$. By the induction hypothesis, $\Psi^{-1}(A)$ is G_δ-dense in $H(\Omega)$. Let $\omega \in \Psi^{-1}(A)$ and let $g \in H(\Omega)$ be such that $g|_{\Omega_j}$ is a polynomial of degree at most $n-2$, for all $j \in \zeta$. We shall show that $\omega + g \in \Psi^{-1}(A)$.

Indeed we have $\Psi(\omega) \in A$ and we have to show that $\Psi(\omega + g) = \Psi(\omega) + \Psi(g)$ belongs also to A. For this it suffices to remark that $\Psi(g)|_{\Omega_j}$ is a polynomial of degree at most $n-1$, for each $j \in J$. This is true, because $\Psi(g)|_{\Omega_j}$ is an antiderivative of $g|_{\Omega_j}$ and $g|_{\Omega_j}$ is a polynomial of degree at most $n-2$.

By the induction hypothesis the set $\Psi^{-(n-1)}(\Psi^{-1}(A))$ is G_δ-dense in $H(\Omega)$. But this set is exactly $\Psi^{-n}(A)$, which completes the proof. ∎

PROPOSITION 2.13. *Let $\Omega \subset \mathbb{C}$ be a simply connected open set and let A be a G_δ-dense subset of $H(\Omega)$. We assume that for every $h \in H(\Omega)$ such that* *the restriction of h to each component Ω_j, $j \in J$ of Ω is a polynomial P_j with* $\sup_{j \in J} \deg P_j < +\infty$, *and for every $f \in A$, it follows that $f+h$ belongs to A also. Under these assumptions, for each $k \in Z$ the set $A_k = \{f \in H(\Omega) : f^{(k)} \in A\}$ is G_δ-dense in $H(\Omega)$, where for $k \geqslant 0$ $f^{(k)}$ denotes the kth derivative of f, and for $k<0$ $f^{(k)} = \Psi^{|k|}(f)$ is the kth antiderivative of f defined above. Moreover the set $\bigcap_{k \in Z} A_k$ is G_δ-dense in $H(\Omega)$.*

Proof. It suffices to combine the previous Lemmas 2.11 and 2.12.

3. TWO GENERIC PROPERTIES OF HOLOMORPHIC FUNCTIONS

Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, be a simply connected domain. Let V be a connected open neighborhood of $\mathbb{C} \backslash \Omega$ and let $A = \{a_{nk}(z)\}_{n,k=0}^{\infty}$ be an infinite array of analytic functions in V. A is called admissible if it satisfies the following three properties:

(i) For every n the set $\{k : a_{nk}(z) \not\equiv 0 \text{ in } V\}$ is finite; that is, there exists $k(n)$ such that $a_{nk}(z) \equiv 0$ for $k > k(n)$.

(ii) For each k we have $a_{nk}(z) \to 0$, as $n \to +\infty$, uniformly on compact subsets of V.

(iii) For every compact subset K of V there exists $M = M(K) \in (1, +\infty)$ such that

$$\frac{1}{M} \leqslant \left| \sum_{k=0}^{k(n)} a_{nk}(z) \right| \leqslant M \text{ for every } n \text{ and every } z \in K.$$

We notice that $k(n) \to +\infty$, as $n \to +\infty$

For $f \in H(\Omega)$ and for $\zeta \in \Omega$ we consider the Taylor development $f(z) = \sum_{v=0}^{\infty} b_v(f, \zeta) \cdot (z-\zeta)^v$ and let $S_k(f, \zeta)(z) = \sum_{v=0}^{k} b_v(f, \zeta) \cdot (z-\zeta)^v$ denote its partial sums.

Finally we set $A_n(f, \zeta)(z) = \sum_{k=0}^{k(n)} a_{nk}(z)\, S_k(f, \zeta)(z)$ for $n = 0, 1, 2, \ldots$ and $z \in V$. Thus A provides a summability method for the power series expansion of f around $\zeta \in \Omega$, which varies holomorphically as z varies in $V \supset C \backslash \Omega$. This summability method is not necessarily regular, because we have not assumed $\sum_k |a_{nk}|$ to be uniformly bounded.

Examples of admissible matrices are the following:

(1) The matrix $a_{nk} = 0$ for $n \neq k$ and $a_{nn} = 1$. Then $A_n(f, \zeta)(z) = S_n(f, \zeta)(z)$.

(2) The matrix $a_{nk} = \frac{1}{n+1}$ for $0 \leqslant k \leqslant n$ and $a_{nk} = 0$ for $k > n$. Then $A_n(f, \zeta)(z) = \sigma_n^1(f, \zeta)(z)$ are the $(C, 1)$ means of the Taylor development of

f with center ζ. More generally, there are admissible matrices A^{ℓ}, such that $A_n^{\ell}(f, \zeta)(z) = \sigma_n^{\ell}(f, \zeta)(z)$ are the (C, ℓ) means of the Taylor development of f with center ζ.

(3) Let V be an open set such that $\Omega^c \subset V \neq \mathbb{C}$. Consider arbitrary holomorphic functions $g_{nk}\colon V \to \mathbb{C}$, such that $|g_{nk}(z)| \leqslant \frac{1}{2}$ for all $z \in V$. We set $a_{nk}(z) = \frac{1}{n+1}[1 + g_{nk}(z)]$ for $0 \leqslant k \leqslant n$ and $a_{nk} = 0$ for $k > n$. Then $A = (a_{nk}(z))_{n,k=0}^{\infty}$ is an admissible matrix.

(4) An example of an admissible matrix A which is not regular will be given in Section 5 below.

Definition 3.1. Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ be a simply connected domain and let A be an admissible matrix, as above. Let S be an infinite subset of $\{0, 1, 2, ...\}$ and let $\zeta \in \Omega$. A function $f \in H(\Omega)$ belongs to the class $U(\Omega, A, S, \zeta)$, iff, for every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \varnothing$ with K^c connected and every function $g\colon K \to \mathbb{C}$ continuous on K and holomorphic in K^0, there exists a sequence $n_r \in S$, $r = 1, 2, ...$ such that $A_{n_r}(f, \zeta)(z) \to g(z)$ uniformly on K as $r \to +\infty$.

Definition 3.2. Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, be a simply connected domain and A an admissible matrix, as above. Let S be an infinite subset of $\{0, 1, 2, ...\}$ and $L \subset \Omega$ be compact. A function $f \in H(\Omega)$ belongs to the class $U(\Omega, A, S, L)$, iff, for every compact set $K \in \mathbb{C}$, $K \cap \Omega = \varnothing$ with K^c connected and every function $g\colon K \to \mathbb{C}$ continuous on K and holomorphic in K^0, there exists a sequence $n_r \in S$, $r = 1, 2, ...$, such that

$$\sup_{\zeta \in L} \sup_{z \in K} |A_{n_r}(f, \zeta)(z) - g(z)| \to 0, \qquad \text{as} \quad r \to +\infty.$$

Definition 3.3. Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, be a simply connected domain and A an admissible matrix, as above. Let S be an infinite subset of $\{0, 1, 2, ...\}$. A function $f \in H(\Omega)$ belongs to the class $U(\Omega, A, S)$, iff, for every compact set $K \in \mathbb{C}$, $K \cap \Omega = \varnothing$ with K^c connected and every function $g\colon K \to \mathbb{C}$ continuous on K and holomorphic in K^0, there exists a sequence $n_r \in S$, $r = 1, 2, ...$, such that for every compact set $L \subset \Omega$ we have

$$\sup_{\zeta \in L} \sup_{z \in K} |A_{n_r}(f, \zeta)(z) - g(z)| \to 0, \qquad \text{as} \quad r \to +\infty.$$

Remark. If $\zeta \in L$, $L \subset \Omega$, then we obviously have $U(\Omega, A, S) \subset U(\Omega, A, S, L) \subset U(\Omega, A, S, \zeta)$.

Remark. We notice that in the above definitions the compact set K is allowed to contain pieces of the boundary $\partial\Omega$; thus the approximation holds on the points of the boundary also. This is an essential difference from the results of Chui and Parnes [3] and Luh [13, 14]. The fact that the approximation is uniform with respect to the center ζ, when ζ varies on compact sets $L \subset \Omega$, seems to be a new element.

Remark. It is easily seen that in the above definitions we may always arrange so that n_r is strictly increasing. Indeed, if g is different from all $A_n(f, \zeta)$, $n \in S$, then the sequence n_r, $r = 1, 2, ...$, cannot have bounded subsequence and therefore we obtain the result passing to a subsequence. Now if $g = A_{n_0}(f, \zeta)$ for some $n_0 \in S$, then for appropriate $\varepsilon > 0$, sufficiently small, the function $g + \varepsilon$ is different from all $A_n(f, \zeta)$, $n \in S$, and we can approximate this new function by $A_{n_r}(f, \zeta)$ with large $n_r \in S$. This gives the result.

We will prove now the following theorem, which provides generic properties of holomorphic functions.

THEOREM 3.4. *Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, be a simply connected domain and A an admissible matrix, as above. Let S be an infinite subset of $\{0, 1, 2, ...\}$, $\zeta \in \Omega$ and $L \subset \Omega$ compact.*

Then the classes $U(\Omega, A, S, \zeta)$, $U(\Omega, A, S, L)$ and $U(\Omega, A, S)$ are G_δ-dense in $H(\Omega)$ endowed with the topology of uniform convergence on compacta. In particular these classes are non-void.

According to Lemma 2.1 we can fix a sequence $K_m \subset \mathbb{C}$, $m = 1, 2, ...$, of compact sets, $K_m \cap \Omega = \varnothing$ with K_m^c connected, such that for every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \varnothing$ with K^c connected, there exists m such that $K \subset K_m$. We also fix an enumeration f_j, $j = 1, 2, ...$ of the polynomials with coefficients in $Q + iQ$. For $\zeta \in \Omega$, m, j, $s = 1, 2, ...$, and $n = 0, 1, 2, ...$ we consider the set

$$E(\Omega, A, \zeta, K_m, f_j, s, n) = \left\{ g \in H(\Omega) : \sup_{z \in K_m} |A_n(g, \zeta)(z) - f_j(z)| < \frac{1}{s} \right\}.$$

LEMMA 3.5. *Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, be a simply connected domain. Let S be an infinite subset of $\{0, 1, 2, ...\}$, $\zeta \in \Omega$ and $L \subset \Omega$ compact. Let also L_ρ, $\rho = 1, 2, ...$, be an exhaustive sequence of compact sets in Ω and let A be an admissable matrix in Ω. Then we have*

(i) $U(\Omega, A, S, \zeta) = \bigcap_m \bigcap_j \bigcap_s \bigcup_{n \in S} E(\Omega, A, \zeta, K_m, f_j, s, n)$

(ii) $U(\Omega, A, S, L) = \bigcap_m \bigcap_j \bigcap_s \bigcup_{n \in S} [\bigcap_{\zeta \in L} E(\Omega, A, \zeta, K_m, f_j, s, n)]$

(iii) $U(\Omega, A, S) = \bigcap_{\rho=1}^{\infty} U(\Omega, A, S, L_\rho)$.

Proof. (i) is a special case of (ii) for $L=\{\zeta\}$. We prove (ii). Obviously

$$U(\Omega, A, S, L) \subset \bigcap_m \bigcap_j \bigcap_s \bigcup_{n\in S} \left[\bigcap_{\zeta\in L} E(\Omega, A, \zeta, K_m, f_j, s, n) \right].$$

Conversely let $f\in \bigcap_m \bigcap_j \bigcap_S \bigcup_{n\in S} [\bigcap_{\zeta\in L} E(\Omega, A, \zeta, K_m, f_j, s, n)]$, $K\subset\mathbb{C}$ compact, $K\cap\Omega=\varnothing$ with K^c connected, and let $h: K\to\mathbb{C}$ be continuous on K and holomorphic in K^0. Let also $\varepsilon>0$. By Mergelyan's theorem [20], there exists f_j, $j\in\{1, 2, ...\}$, such that $\sup_{z\in K} |h(z)-f_j(z)|<\frac{\varepsilon}{2}$. By Lemma 2.1 there exists $m=1, 2, ...$ such that $K_m\subset K$. Then for any s with $\frac{1}{s}<\frac{\varepsilon}{2}$, we have $f\in\bigcup_{n\in S} [\bigcap_{\zeta\in L} E(\Omega, A, \zeta, K_m, f_j, s, n)]$. Thus, there exists $n\in S$ such that $\sup_{\zeta\in L} \sup_{z\in K_m} |A_n(f,\zeta)(z)-f_j(z)|\leqslant\frac{1}{s}<\frac{\varepsilon}{2}$. Since $K\subset K_m$ we also have $\sup_{\zeta\in L} \sup_{z\in K} |A_n(f,\zeta)(z)-f_j(z)|<\frac{\varepsilon}{2}$. It follows that $\sup_{\zeta\in L}$ $\sup_{z\in K_m} |A_n(f,\zeta)(z)-h(z)|<\varepsilon$. Thus $f\in U(\Omega, A, S, L)$ and the proof of (ii) is complete.

Now we prove (iii). Obviously $U(\Omega, A, S)\subset\bigcap_{\rho=1}^{\infty} U(\Omega, A, S, L_\rho)$. Conversely let $f\in\bigcap_{\rho=1}^{\infty} U(\Omega, A, S, L_\rho)$. Let $K\subset\mathbb{C}$ compact, $K\cap\Omega=\varnothing$ with K^c connected and $h: K\to\mathbb{C}$ continuous on K and holomorphic in K^o. Then for every $\rho=1, 2, ...$ we can find $n_\rho\in S$, such that $\sup_{\zeta\in L_\rho} \sup_{z\in K} |A_{n_\rho}(f,\zeta)(z)-h(z)|<\frac{1}{\rho}$ for all $\rho=1, 2, ...$. Let $L\subset\Omega$ compact; then there exists ρ_o such that $L\subset L_{\rho_0}\subset L_{\rho_{0+1}}\subset ...$. It follows that for $\rho\geqslant\rho_0$ we have $\sup_{\zeta\in L} \sup_{z\in K} |A_{n_\rho}(f,\zeta)(z)-h(z)|<\frac{1}{\rho}\to 0$, as $\rho\to+\infty$. This shows that $f\in U(\Omega, A, S)$ and the proof of the lemma is complete. ∎

Since $H(\Omega)$ is a complete metrizable space, we have Baire's theorem in our disposal. In view of Lemma 3.5, the result of Theorem 3.4 will follow, if we prove that $\bigcap_{\zeta\in L} E(\Omega, A, \zeta, K_m, f_j, s, n)$ is open in $H(\Omega)$ and that $\bigcup_{n\in S} [\bigcap_{\zeta\in L} E(\Omega, A, \zeta, K_m, f_j, s, n)]$ is dense in $H(\Omega)$.

LEMMA 3.6. *With the above notations and assumptions, for every compact set $L\subset\Omega$ the set $\bigcap_{\zeta\in L} E(\Omega, A, \zeta, K_m, f_j, s, n)$ is open in $H(\Omega)$.*

Proof. Let L_ρ, $\rho=1, 2, ...$ be an exhaustive sequence of compact sets in Ω. Then there exists ρ such that $L\subset L_\rho\subset L^o_{\rho+1}$. We set $2\tau=$ *dist* $(L_\rho, \mathbb{C}\backslash L^o_{\rho+1})>0$, $C=\sup\{|z-\zeta|: \zeta\in L, z\in K_m\}<+\infty$ and $\sup_{z\in K_m}$ $\sum_{k=0}^{k(n)} |a_{nk}(z)|=B<+\infty$. Let $f\in\bigcap_{\zeta\in L} E(\Omega, A, \zeta, K_m, f_j, s, n)$. By continuity on the compact set $K_m\times L$ we obtain $\sup_{\zeta\in L} \sup_{z\in K_m} |A_n(f,\zeta)(z)$ $-f_j(z)|<\frac{1}{s}$. Choose $\delta>0$, such that

$$\left(\sum_{k=0}^{k(n)} \tau^{-k} C^k\right) B\delta + \sup_{\zeta\in L} \sup_{z\in K_m} |A_n(f,\zeta)(z)-f_j(z)| < \frac{1}{s}.$$

Suppose that $g \in H(\Omega)$ satisfies $\sup_{z \in L_{p+1}} |f(z) - g(z)| < \delta$; we will show that $g \in \bigcap_{\zeta \in L} E(\Omega, A, \zeta, K_m, f_j, s, n)$. This will prove the lemma. We set $F(z) = f(z) - g(z) = \sum_{k=0}^{\infty} b_k(\zeta)(z-\zeta)^k$ $(\zeta \in L)$.

By the Cauchy estimates we have $|b_k(\zeta)| \leqslant \tau^{-k} \sup_{z \in L_{p+1}} |F(z)| \leqslant \tau^{-k}\delta$. Thus, for $v \leqslant k(n)$, $\zeta \in L$, and $z \in K_m$, we get

$$|S_v(F, \zeta)(z)| \leqslant \sum_{k=0}^{v} |b_k(\zeta)|\, |z-\zeta|^k \leqslant \sum_{k=0}^{k(n)} \tau^{-k} C^k \cdot \delta.$$

Hence

$$|A_n(f, \zeta)(z) - A_n(g, \zeta)(z)| \leqslant \sum_{v=0}^{k(n)} |a_{nv}(z)| \cdot |S_v(F, \zeta)(z)| \leqslant B \cdot \sum_{k=0}^{k(n)} \tau^{-k} C^k \cdot \delta.$$

It follows that, for $\zeta \in L$ and $z \in K_m$ we have

$$|A_n(g, \zeta)(z) - f_j(z)| \leqslant |A_n(g, \zeta)(z) - A_n(f, \zeta)(z)| + |A_n(f, \zeta)(z) - f_j(z)| < \frac{1}{s}.$$

By continuity on the compact set K_m we obtain $\sup_{z \in K_m} |A_n(g, \zeta)(z) - f_j(z)| < \frac{1}{s}$ for all $\zeta \in L$. Thus $g \in \bigcap_{\zeta \in L} E(\Omega, A, \zeta, K_m, f_j, s, n)$ and the proof is complete. ▌

LEMMA 3.7. *With the above notations and assumptions for every compact set $L \subset \Omega$ the set $\bigcup_{n \in S} [\bigcap_{\zeta \in L} E(\Omega, A, \zeta, K_m, f_j, s, n)]$ is dense in $H(\Omega)$.*

Proof. Let $f \in H(\Omega)$, $\tilde{L} \subset \Omega$ compact with $\tilde{L}^c$ connected, $\varepsilon > 0$ and $V_\varepsilon(f) = \{g \in H(\Omega) : \sup_{w \in \tilde{L}} |f(w) - g(w)| < \varepsilon\}$. It suffices to show that $\bigcup_{n \in S} [\bigcap_{\zeta \in L} E(\Omega, A, \zeta, K_m, f_j, s, n)]$ intersects $V_\varepsilon(f)$.

Since A is admissible, there are an open set W, $K_m \subset W$ with $\bar{W} \subset V$ compact and a constant $M \in (0, +\infty)$, such that $\frac{1}{M} \leqslant |\sum_{k=0}^{k(n)} a_{nk}(z)| \leqslant M$ for all $z \in W$. By normal families there exist $h \in H(W)$ and a sequence $n_r \in S$, $r = 1, 2, ...,$ such that $\sum_{k=0}^{k(n_r)} a_{n_r k}(z) \to h(z)$ uniformly on K_m as $r \to +\infty$. Obviously $\frac{1}{M} \leqslant |h(z)| \leqslant M$ on K_m.

The compact sets $\tilde{L}$ and K_m are disjoint and they have connected complements. Therefore, the same holds for the compact set $\tilde{L} \cup K_m$. By Mergelyan's theorem we can find a polynomial g such that $\sup_{w \in \tilde{L}} |f(w) - g(w)| < \varepsilon$ and $\sup_{z \in K_m} |g(z) - (f_j(z)/h(z))| < \frac{1}{2Ms}$. We chose $n_0 \in \{0, 1, 2, ...\}$ with $n_0 \geqslant \deg g$, and let $\varepsilon_n = \sup_{z \in K_m} \sum_{k=0}^{n_0} |a_{nk}(z)|$ and $C = \sup_{0 \leqslant k \leqslant n_0} \sup_{\zeta \in L} \sup_{z \in K_m} |S_k(g, \zeta)(z)| < +\infty$. Because A is admissible we have $\varepsilon_n \to 0$, as $n \to +\infty$. Also let $\delta_r = \sup_{z \in K_m} |\sum_{k=0}^{k(n_r)} a_{n_r k}(z) - h(z)|$, where $\delta_r \to 0$, as $r \to +\infty$.

Since $\deg g \leqslant n_0$ we have $S_n(g, \zeta)(z) = g(z)$ for all $n \geqslant n_0$ and $z, \zeta \in \mathbb{C}$. Therefore,

$$\begin{aligned} A_n(g, \zeta)(z) &= \sum_{k=0}^{n_0} a_{nk}(z)\, S_k(g, \zeta)(z) + g(z) \sum_{k=n_0+1}^{k(n)} a_{nk}(z) \\ &= \sum_{k=0}^{n_0} a_{nk}(z)\, S_k(g, \zeta)(z) + g(z)\, h(z) + g(z) \left[\sum_{k=0}^{k(n)} a_{kn}(z) - h(z) \right] \\ &\quad - g(z) \sum_{k=0}^{n_0} a_{nk}(z). \end{aligned}$$

It follows that

$$\begin{aligned} |A_{n_r}(g, \zeta)(z) - f_j(z)| &\leqslant \sum_{k=0}^{n_0} |a_{n_r k}(z)| \{ |S_k(g, \zeta)(z)| + |g(z)| \} \\ &\quad + |g(z)| \left| \sum_{k=0}^{k(n_r)} a_{n_r k}(z) - h(z) \right| + |h(z)| \left| g(z) - \frac{f_j(z)}{h(z)} \right| \\ &\leqslant 2C\varepsilon_{n_r} + C\delta_r + M \cdot \frac{1}{2Ms} = C(2\varepsilon_{n_r} + \delta_r) + \frac{1}{2s}, \end{aligned}$$

for $z \in K_m$ and $\zeta \in L$. We may choose r sufficiently large, so that $n_r \geqslant n_0$ and $C(2\varepsilon_{n_r} + \delta_r) < \frac{1}{2s}$. Therefore, by continuity on the compact set K_m we get $\sup_{z \in K_m} |A_{n_r}(g, \zeta)(z) - f_j(z)| < \frac{1}{s}$ for all $\zeta \in L$. Therefore g belongs to the intersection of $V_\varepsilon(f)$ with the set $\bigcap_{\zeta \in L} E(\Omega, A, \zeta, K_m, f_j, s, n_r)$, where $n_r \in S$. This completes the proof of the lemma and of Theorem 3.4. ∎

Remark. By Baire's Theorem a denumerable intersection of G_δ-dense subsets of a complete metrizable space is again G_δ-dense. Thus, if we consider a denumerable set of admissible matrices, the intersection of the corresponding classes of functions is non–empty. This implies that there exists a holomorphic function $f \in H(\Omega)$, such that, for every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \varnothing$ with K^c connected and every function $h: K \to \mathbb{C}$ continuous on K and holomorphic in K^o, h can be uniformly approximated by the (C, k) means of the Taylor development of f with some center $\zeta \in \Omega$ and this for every $k = 0, 1, 2, \ldots$. Furthermore, by modification of our proof the approximating sequence λ_n can be chosen the same for all k, as we will see in Section 4.

Remark. The class $U(\Omega, A, S)$ of Definition 3.3 can be considered even when Ω is not simply connected. For a domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ we denote by $U(\Omega)$ the class $U(\Omega, A, S)$, where $S = \{0, 1, 2, \ldots\}$ and $A = (a_{nk})_{n, k=0}^{\infty}$ with $a_{nn} = 1$ and $a_{nk} = 0$ for $n \neq k$ (see also Definition 1.3 in the introduction).

The following Proposition shows that $U(\Omega)=\emptyset$ when Ω is not simply connected. This explains why we always assume that Ω is simply connected when we discuss these classes.

PROPOSITION 3.8. *If $\Omega\subset\mathbb{C}$ is a domain such that Ω^c has at least a bounded connected component then the class $U(\Omega)$ is empty.*

Proof. We may assume that 0 belongs to a bounded connected component of Ω^c. Then it is easy to construct a cycle Γ in Ω such that the index of 0 with respect to Γ is equal to 1. That is $\frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{\zeta}=1$. Now for any $f\in H(\Omega)$ and any $n\geqslant 0$ we have

$$S_n(f,\zeta)(0)=\sum_{k=0}^{n}\frac{f^{(k)}(\zeta)}{k!}(-\zeta)^k$$

and therefore

$$\frac{1}{2\pi i}\int_\Gamma \frac{S_n(f,\zeta)(0)}{\zeta}\,d\zeta=\sum_{k=0}^{n}\frac{(-1)^k}{k!}\frac{1}{2\pi i}\int_\Gamma f^{(k)}(\zeta)\,\zeta^{k-1}\,d\zeta.$$

Since Γ is a cycle in Ω we have: $\int_\Gamma f'(\zeta)\,d\zeta=0$, $\int_\Gamma f''(\zeta)\,\zeta\,d\zeta=\int_\Gamma[(f'(\zeta)\,\zeta)'-f'(\zeta)]\,d\zeta=0$ and by an easy induction $\int_\Gamma f^{(k)}(\zeta)\,\zeta^{k-1}\,d\zeta=0$ for every $k\geqslant 1$. Therefore

$$\frac{1}{2\pi i}\int_\Gamma \frac{S_n(f,\zeta)(0)}{\zeta}\,d\zeta=\frac{1}{2\pi i}\int_\Gamma\frac{f(\zeta)}{\zeta}\,d\zeta=A\in\mathbb{C},$$

where A depends only on f and not on n. Hence it follows that $S_n(f,\zeta)(z)$ cannot approximate the constant function $A+1$ on the compact set $K=\{0\}\subseteq\Omega^c$ uniformly for $\zeta\in\Gamma^*$ and so $U(\Omega)$ is empty. ∎

Theorem 3.4 provides a generic property of holomorphic functions. Now we proceed to establish another such property.

Let Ω be any simply connected domain, $\Omega\subset\mathbb{C}$. Let L and K be two non-empty compact subsets of Ω and let S be any infinite subset of $\{0, 1, 2, 3, ...\}$.

DEFINITION 3.9. A function $f\in H(\Omega)$ belongs to the class $B(\Omega, L, K, S)$, iff there exists a sequence $\lambda_n\in S$, $n=1, 2, ...,$ such that $\sup_{\zeta\in L}\sup_{z\in K}|S_{\lambda_n}(f,\zeta)(z)-f(z)|\to 0$ as $n\to\infty$.

For $\zeta\in\Omega$, $n=0, 1, 2, ...$ and $s=1, 2, ...,$ we consider the set $\Gamma(\Omega,\zeta,K,n,s)=\{f\in H(\Omega):\sup_{z\in K}|S_n(f,\zeta)(z)-f(z)|<\frac{1}{s}\}$ and we have the following.

LEMMA 3.10. *With the above notations and assumptions we have* $B(\Omega, L, K, S) = \bigcap_{s=1}^{\infty} \bigcup_{n \in S} [\bigcap_{\zeta \in L} \Gamma(\Omega, \zeta, K, n, s)]$.

Proof. Obviously $B(\Omega, L, K, S) \subset \bigcap_{s=1}^{\infty} \bigcup_{n \in S} [\bigcap_{\zeta \in L} \Gamma(\Omega, \zeta, K, n, s)]$. Conversely, let $f \in \bigcap_{s=1}^{\infty} \bigcup_{n \in S} [\bigcap_{\zeta \in L} \Gamma(\Omega, \zeta, K, n, s)]$. Then, for every $s = 1, 2, ...$ there exists $n_s \in S$ such that $\sup_{z \in K} |S_{n_s}(f, \zeta)(z) - f(z)| < \frac{1}{s}$ for all $\zeta \in L$. By continuity on the compact set $L \times K$ we obtain $\sup_{\zeta \in L} \sup_{z \in K} |S_{n_s}(f, \zeta)(z) - f(z)| < \frac{1}{s} \to 0$ as $s \to +\infty$. Therefore, $f \in B(\Omega, L, K, S)$. This completes the proof of the lemma. ∎

PROPOSITION 3.11. *The set* $\bigcap_{\zeta \in L} \Gamma(\Omega, \zeta, K, n, s)$ *is open in* $H(\Omega)$.

Proof. Let $f \in \bigcap_{\zeta \in L} \Gamma(\Omega, \zeta, K, n, s)$; then by continuity on the compact set $L \times K$ we have $\sup_{\zeta \in L} \sup_{z \in K} |S_n(f, \zeta)(z) - f(z)| = \beta < \frac{1}{s}$. Let L_ρ, $\rho = 1, 2, ...,$ be an exhaustive sequence of compact sets in Ω. Then $L \subset L_\rho \subset (L_{\rho+1})^o$ for some ρ. Let $2\tau = dist(L, (L^o_{\rho+1})^c) > 0$ and $M = \sup_{\zeta \in L} \sup_{z \in K} |\zeta - z| < +\infty$. We choose $a > 0$ such that $a[1 + \sum_{k=0}^{n} \frac{M^k}{\tau^k}] + \beta < \frac{1}{s}$.

Let $g \in H(\Omega)$ such that $\sup_{w \in K \cup L_{\rho+1}} |f(w) - g(w)| < a$. For $\zeta \in L$ we write $f(w) - g(w) = \sum_{k=0}^{\infty} b_k(\zeta)(w - \zeta)^k$. An application of the Cauchy estimates yields $|b_k(\zeta)| < \frac{a}{\tau^k}$. It follows that for $z \in K$ and $\zeta \in L$ we have

$$|S_n(g, \zeta)(z) - S_n(f, \zeta)(z)| < \left(\sum_{k=0}^{n} \frac{M^k}{\tau^k} \right) a \qquad \text{and} \qquad |f(z) - g(z)| < a.$$

Thus, $\sup_{z \in K} |S_n(g, \zeta)(z) - g(z)| < a\,[1 + \sum_{k=0}^{n} \frac{M^k}{\tau^k}] + \beta < \frac{1}{s}$, for all $\zeta \in L$. This shows that $g \in \bigcap_{\zeta \in L} \Gamma(\Omega, \zeta, K, n, s)$ and the proof is complete. ∎

PROPOSITION 3.12. *The set* $\bigcup_{n \in S} [\bigcap_{\zeta \in L} \Gamma(\Omega, \zeta, K, n, s)]$ *is dense in* $H(\Omega)$.

Proof. Let $f \in H(\Omega)$, let $\varepsilon > 0$, and let F be a compact subset of Ω. Since Ω is simply connected, we may assume that F^c is connected. It suffices to find $g \in H(\Omega)$ and $n \in S$ such that $g \in \bigcap_{\zeta \in L} \Gamma(\Omega, \zeta, K, n, s)$ and $\sup_{z \in F} |f(z) - g(z)| < \varepsilon$.

By Mergelyan's theorem we can find a polynomial g such that $\sup_{z \in F} |f(z) - g(z)| < \varepsilon$. We also choose $n \in S$ such that $n \geqslant \deg g$. Then, for every $\zeta \in \mathbb{C}$ we have $S_n(g, \zeta) \equiv g$ and therefore $\sup_{z \in F} |S_n(g, \zeta)(z) - g(z)| = 0 < \frac{1}{s}$ for all $\zeta \in L$. This gives the result. ∎

PROPOSITION 3.13. *With the previous notations and assumptions we consider* L_ρ, K_τ, $\rho = 1, 2, ...,$ *and* $\tau = 1, 2, ...,$ *two exhaustive sequences of compact sets in* Ω *and we have*:

(i) *The set* $B(\Omega, L, K, S)$ *is* G_δ*-dense in* $H(\Omega)$.

(ii) *The set* $\bigcap_{\tau=1}^{\infty} B(\Omega, L, K_\tau, S)$ *is* G_δ*-dense in* $H(\Omega)$.

(iii) *The set* $\bigcap_{\rho=1}^{\infty} \bigcap_{\tau=1}^{\infty} B(\Omega, L_\rho, K_\tau, S)$ *is* G_δ*-dense in* $H(\Omega)$.

Proof. Since $H(\Omega)$ is metrizable complete we have Baire's Theorem at our disposal. This combined with Lemma 3.10, Proposition 3.11, and Proposition 3.12 gives (i). Statement (i) combined with Baire's Theorem again implies (ii) and (iii). The proof is complete. ∎

DEFINITION 3.14. Let Ω be a simply connected domain, let $\Omega \subset \mathbb{C}$, and let S be any infinite subset of $\{0, 1, 2, ...\}$. A function $f \in H(\Omega)$ belongs to the class $B(\Omega, S)$ iff there exists a sequence $\lambda_n \in S$, $n = 1, 2, ...$, such that, for every pair of non-empty compact sets $L \subset \Omega$ and $K \subset \Omega$, we have

$$\sup_{\zeta \in L} \sup_{z \in K} |S_{\lambda_n}(f, \zeta)(z) - f(z)| \to 0, \qquad \text{as} \quad n \to +\infty.$$

Remark. In the above definition, it is easily seen that we can arrange so that $\lambda_n < \lambda_{n+1}$ for all $n = 1, 2, ...$.

LEMMA 3.15. *Let* L_ρ, K_τ, $\rho = 1, 2, ...$ *and* $\tau = 1, 2, ...$, *be two exhaustive sequences of compact sets in* Ω. *Then we have*

$$B(\Omega, S) = \bigcap_{\rho=1}^{\infty} \bigcap_{\tau=1}^{\infty} B(\Omega, L_\rho, K_\tau, S) = \bigcap_{\rho=1}^{\infty} B(\Omega, L_\rho, K_\rho, S).$$

Proof. Obviously $B(\Omega, S) \subset \bigcap_\rho \bigcap_\tau B(\Omega, L_\rho, K_\tau, S) \subset \bigcap_\rho B(\Omega, L_\rho, K_\rho, S)$.

Let $f \in \bigcap_\rho B(\Omega, L_\rho, K_\rho, S)$; then for every $\rho = 1, 2, ...$ we can choose $\lambda_\rho \in S$ such that $\sup_{\zeta \in L_\rho} \sup_{z \in K_\rho} |S_{\lambda_\rho}(f, \zeta)(z) - f(z)| < \frac{1}{\rho}$. Thus, we have defined a sequence $\lambda_\rho \in S$, $\rho = 1, 2, ...$.

Let L and K be two non–empty compact subsets of Ω. Then there exists ρ_o, such that $L \subset L_{\rho_o}$ and $K \subset K_{\rho_o}$. Therefore, $L \subset L_\rho$ and $K \subset K_\rho$, for all $\rho \geqslant \rho_o$. It follows that for $\rho \geqslant \rho_o$ we have

$$\sup_{\zeta \in L} \sup_{z \in K} |S_{\lambda_\rho}(f, \zeta)(z) - f(z)| < \frac{1}{\rho} \to +0, \qquad \text{as} \quad \rho \to +\infty.$$

This implies $f \in B(\Omega, S)$ and completes the proof. ∎

Lemma 3.15 combined with Proposition 3.13 implies the following.

THEOREM 3.16. *Let* $\Omega \subset \mathbb{C}$ *be a simply connected domain and* S *any infinite subset of* $\{0, 1, 2, ...\}$. *Then the class* $B(\Omega, S)$ *is* G_δ*-dense in* $H(\Omega)$.

In the case $\Omega \neq \mathbb{C}$ we have already proved that the class $U(A, S)$ (Def. 3.3) is a G_δ-dense set in $H(\Omega)$ (Theorem 3.4). By Baire's Theorem the intersection of two G_δ-dense subsets is again G_δ-dense. Thus, we have the following.

PROPOSITION 3.17. *Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ be a simply connected domain. Let A be an admissible matrix and let S_1, S_2 be two infinite subsets of $\{0, 1, 2, ...\}$. Then there exists a holomorphic function $f \in H(\Omega)$ satisfying the following two properties*:

(i) *There exists a strictly increasing sequence $\lambda_n \in S_1$, $n = 1, 2, ...,$ such that for every pair of non empty compact sets $L \subset \Omega$ and $\tilde{K} \subset \Omega$ we have*

$$\sup_{\zeta \in L} \sup_{z \in \tilde{K}} |S_{\lambda_n}(f, \zeta)(z) - f(z)| \to 0, \qquad as \quad n \to +\infty.$$

(ii) *For every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \varnothing$, with K^c connected and every function $g: K \to \mathbb{C}$, continuous on K and holomorphic in K^o, there exists a strictly increasing sequence $\mu_n \in S_2$, $n = 1, 2, ...,$ such that, for every non-empty compact set $L \subset \Omega$, we have*

$$\sup_{\zeta \in L} \sup_{z \in K} |A_{\mu_n}(f, \zeta)(z) - g(z)| \to 0, \qquad as \quad n \to +\infty.$$

Furthermore, generically all functions f in $H(\Omega)$ satisfy properties (i) *and* (ii).

In the above proposition we do not know if $\sup_{\zeta \in L} \sup_{z \in \tilde{K}} |S_{\mu_n}(f, \zeta)(z) - f(z)| \to 0$ as $n \to +\infty$. In order to obtain this also it suffices to repeat simultaneously the proofs of Theorems 3.4 and 3.16. We do this in the following section.

4. FURTHER GENERIC PROPERTIES AND THE (C, a) MEANS

In this section we prove a more general version of Theorem 3.4, where the matrix A may depend on a parameter and use it to prove Theorem 1.4.

As in Section 3, let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, be a simply connected domain and let V be an open neighborhood of $\mathbb{C} \setminus \Omega$. Also let X be a hemi–compact topological space; that is, there exists a sequence J_ρ, $\rho = 1, 2, ...,$ of compact subsets of X with the property that every compact set $J \subset X$ is contained in some J_ρ, $\rho = 1, 2, ...$ Let $A = (a_{nk}(z, x))_{n,k=0}^{\infty}$ be an infinite array of continuous functions on $V \times X$. Then we say that A is X-admissible if it satisfies the following properties:

(i) For every n there exists $k(n)$ such that $a_{nk}(z, x) = 0$ for every $k > k(n)$, $z \in V$, and $x \in X$.

(ii) For every k we have $a_{nk}(z, x) \to 0$ as $n \to +\infty$, uniformly on compact subsets of $V \times X$.

(iii) There exists a function h analytic in V such that $h(z) \neq 0$ for every $z \in V$, which satisfies $\sum_{k=0}^{k(n)} a_{nk}(z, x) \to h(z)$ as $n \to +\infty$, uniformly on compact subsets of $V \times X$.

For any $f \in H(\Omega)$ we set

$$A_n(f, \zeta, x)(z) = \sum_{k=0}^{k(n)} a_{nk}(z, x)\, S_k(f, \zeta)(z)$$

for $n = 0, 1, 2, ..., \zeta \in \Omega$, $z \in V$, and $x \in X$.

DEFINITION 4.1. A holomorphic function $f \in H(\Omega)$ is said to belong to the class $X(\Omega, A)$ iff the following hold:

For every compact set $K \subset \mathbb{C} \backslash \Omega$ with K^c connected and every function $\phi: K \to \mathbb{C}$ continuous on K and holomorphic in K^o, there exists a sequence λ_n, $n = 1, 2, ...$, of non-negative integers such that, for every non-empty compact sets $L, \tilde{L} \subset \Omega$ and $J \subset X$ we have

(i) $\sup_{\zeta \in L} \sup_{w \in \tilde{L}} |S_{\lambda_n}(f, \zeta)(w) - f(w)| \to 0$ as $n \to +\infty$ and

(ii) $\sup_{x \in J} \sup_{\zeta \in L} \sup_{z \in K} |A_{\lambda_n}(f, \zeta, x)(z) - \phi(z)| \to 0$ as $n \to +\infty$.

Remark. In the above definition we can always arrange that $\lambda_n < \lambda_{n+1}$. If S is an infinite subset of $\{0, 1, 2, ...\}$ we can also define $X(\Omega, A, S)$ by requiring in definition 4.1 that $\lambda_n \in S$. The result we obtain for $X(\Omega, A)$ is also valid for $X(\Omega, A, S)$, with the proof requiring only trivial modifications.

THEOREM 4.3. *If A is X-admissible and Ω is a simply connected domain, $\Omega \neq \mathbb{C}$, then the class $X(\Omega, A)$ is G_δ-dense in $H(\Omega)$.*

Proof. Let L_ρ, $\rho = 1, 2, ...$, be an exhausting sequence of compact subsets of Ω and let J_ρ, $\rho = 1, 2, ...$, be a sequence of compact subsets of X such that every compact set $J \subset X$ is contained in some J_ρ, $\rho = 1, 2, ...$. Because Ω is simply connected, we may assume that L_ρ^c is connected, for every $\rho = 1, 2, ...$.

For m, j, s positive integers, $n = 0, 1, 2, ...$, L, and $\tilde{L}$ compact subsets of Ω, and $J \subset X$ compact, according to the notations of Theorem 3.4, we consider the set

$$\begin{aligned} &Y(\Omega, K_m, L, \tilde{L}, J, f_j, n, s) \\ &\quad = \Big\{ g \in H(\Omega) : \sup_{\zeta \in L} \sup_{w \in \tilde{L}} |S_n(g, \zeta)(w) - g(w)| < \frac{1}{s} \\ &\qquad\qquad \text{and} \quad \sup_{z \in J} \sup_{\zeta \in L} \sup_{z \in K_m} |A_n(g, \zeta, x)(z) - f_j(z)| < \frac{1}{s} \Big\}. \end{aligned}$$

Then, as in Lemma 3.5, we have

$$X(\Omega, A) = \bigcap_{\rho=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{n=0}^{\infty} Y(\Omega, K_m, L_\rho, L_\rho, J_\rho, f_j, n, s)$$

It suffices, therefore, to prove that each $Y(\Omega, K_m, L, \tilde{L}, J, f_j, n, s)$ is open in $H(\Omega)$ and that each $\bigcup_{n=0}^{\infty} Y(\Omega, K_m, L, \tilde{L}, J, f_j, s, n)$ is dense in $H(\Omega)$

For the first let $f \in Y(\Omega, K_m, L, \tilde{L}, J, f_j, n, s)$ and let ρ be such that $L \cup \tilde{L} \subset L_\rho \subset L^o_{\rho+1}$. Since each $a_{nk}(z, x)$ is continuous on the compact set $K_m \times J$, we have $\sup_{z \in K_m} \sup_{x \in J} \sum_{k=0}^{k(n)} |a_{nk}(z, x)| < +\infty$. Therefore, as in Lemma 3.6 and Proposition 3.11, there exists $\delta > 0$ such that, if $g \in H(\Omega)$ satisfies $\sup_{z \in L_{\rho+1}} |f(z) - g(z)| < \delta$, then $\sup_{z \in J} \sup_{\zeta \in L} \sup_{z \in K_m} |A_n(g, \zeta, x)(z) - f_j(z)| < \frac{1}{s}$ and $\sup_{\zeta \in L} \sup_{w \in \tilde{L}} |S_n(g, \zeta)(w) - g(w)| < \frac{1}{s}$.

For the second it suffices to combine the proofs of Lemma 3.7 and Proposition 3.12, since the $h(z)$ in the proof of Lemma 3.7 is provided by condition (iii) of the X-admissibility of the matrix A, and conditions (ii) and (iii) hold uniformly for $x \in J$.

This completes the proof of the theorem. ∎

Using this theorem we can prove Theorem 1.4 as follows:

Proof of Theorem 1.4. Let $X = (-1, +\infty)$ and for $a \in X$, $a \neq 0$ we define

$$b_{nk}(z, a) = b_{nk}(a) = \frac{\binom{n-k+a-1}{n-k}}{\binom{n+a}{n}}$$

if $0 \leqslant k \leqslant n$ and $b_{nk}(a) = 0$ if $k > n$. By continuity we set $b_{nn}(0) = 1$ and $b_{nk}(0) = 0$ for $k \neq n$. Then for the matrix $B = (b_{nk}(a))_{n, k=0}$ we have $B_n(f, \zeta, a)(z) = \sigma_n^a(f, \zeta)(z)$ [21, Vol. I, pp. 76–77]. Therefore to prove Theorem 1.4 it suffices, by Theorem 4.2 and the remark before it, to prove that the matrix B is X-admissible. Obviously $b_{nk}(a)$ are continuous in $(-1, +\infty)$. Condition (i) is obvious with $k(n) = n$, whereas condition (iii) follows with $h(z) = 1$, by $\sum_{k=0}^{n} b_{nk}(a) = 1$ for every $a > -1$. Hence it remains to prove that for each $k \geqslant 0$ we have

$$\lim_{n \to \infty} b_{nk}(a) = 0$$

uniformly on compact subsets of $(-1, +\infty)$.

This follows easily because for $-1 < -1+\varepsilon \leqslant a \leqslant M < +\infty$ and $n > k+1$ we have

$$|b_{nk}(a)| = \left|\frac{a(a+1)\cdots(a+n-k-1)}{(n-k)!} \cdot \frac{n!}{(a+1)\cdots(a+n)}\right|$$

$$= \left|\frac{a\cdot(n-k+1)\cdots n}{(a+n-k)\cdots(a+n)}\right| \leqslant M \cdot \frac{n^k}{(n-k-1)^{k+1}} \to 0, \qquad \text{as} \quad n\to\infty. \tag{4.2}$$

This completes the proof of Theorem 1.4 ∎

Theorem 4.2 has also the following application:

THEOREM 4.3. *Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ be a simply connected domain and let $g_1, g_2, \ldots,$ be a sequence of holomorphic functions in Ω such that for each $k = 1, 2, \ldots,$ the set $\{z \in \Omega : g_k(z) = 0\}$ has no accumulation points in $\mathbb{C}$. Then there exist functions $u, u_1, u_2, \ldots$ all belonging in the class $U(\Omega)$, such that $g_k = \frac{u_k}{u}$ on Ω for all $k = 1, 2, \ldots$.*

Proof. First let h be an entire function with no zeros outside Ω. We consider the class

$$U_h(\Omega) = \{f \in H(\Omega) : hf \in U(\Omega)\}.$$

We shall show that $U_h(\Omega)$ is G_δ-dense subset of $H(\Omega)$.

Let $f \in H(\Omega)$ and $\zeta \in \Omega$. We write

$$h(z) = \sum_{k=0}^{\infty} a_k(\zeta)(z-\zeta)^k \qquad \text{and} \qquad f(z) = \sum_{k=0}^{\infty} b_k(\zeta)(z-\zeta)^k.$$

Therefore, $hf(z) = \sum_{m=0}^{\infty} [\sum_{k=0}^{m} a_k(\zeta)\, b_{m-k}(\zeta)](z-\zeta)^m$ and

$$S_n(hf,\zeta)(z) = \sum_{m=0}^{\infty} \left[\sum_{k=0}^{m} a_k(\zeta)\, b_{m-k}(\zeta)\right](z-\zeta)^m$$

$$= \sum_{k=0}^{n} a_k(\zeta) \left[\sum_{m=k}^{n} b_{m-k}(\zeta)(z-\zeta)^{m-k}\right](z-\zeta)^m$$

$$= \sum_{k=0}^{n} a_k(\zeta)(z-\zeta)^k\, S_{n-k}(f,\zeta)(z). \tag{4.3}$$

We set $X=\Omega$ and $V=\mathbb{C}\setminus\{z\in\mathbb{C}:h(z)=0\}$, which is an open and connected neighborhood of Ω^c. We define the matrix A on $V\times\Omega$ by

$$a_{nk}(z,\zeta)=a_{n-k}(\zeta)(z-\zeta)^{n-k} \qquad \text{for} \quad 0\leqslant k\leqslant n$$

and $a_{nk}(z,\zeta)=0$ for $k>n$ ($z\in V$, $\zeta\in X=\Omega$).

Then $A_n(f,\zeta,\zeta)(z)=S_n(hf,\zeta)(z)$ and hence to complete the proof, it suffices to show that the matrix A is Ω-admissible.

Condition (i) is obvious. Since h is entire the Cauchy estimates imply that for each $k\geqslant 0$ we have $\lim_{n\to\infty} a_{n-k}(\zeta)(z-\zeta)^{n-k}=0$ uniformly on compact subsets of $\mathbb{C}\times\mathbb{C}$. Thus condition (ii) also holds.

Finally,

$$\begin{aligned}\sum_{k=0}^{n} a_{nk}(z,\zeta)&=\sum_{k=0}^{n} a_{n-k}(\zeta)(z-\zeta)^{n-k}=S_n(h,\zeta)(z)\\ &=h(z)-\sum_{r=n+1}^{\infty} a_r(\zeta)(z-\zeta)^r\to h(z), \qquad \text{as} \quad n\to+\infty, \end{aligned} \tag{4.4}$$

uniformly on compact subsets of $\mathbb{C}\times\mathbb{C}$, again by the Cauchy estimates. Moreover $h(z)\neq 0$ on V. Hence A satisfies condition (iii) also. Therefore for each such h the set $U_h(\Omega)$ defined at the beginning of the proof is G_δ-dense in $H(\Omega)$.

Now since for each $k=1,2,...,$ the zero set of g_k has no accumulation points in $\mathbb{C}$ by Weierstrass factorization theorem there exist entire functions h_k for $h=1,2,...,$ such that each h_k has the same zero set with g_k and with the same multiplicities. Hence we can write

$$g_k=h_k w_k$$

for $k=1,2,...,$ and each w_k is a function defined and holomorphic in Ω that has no zeros. Then for each k the mappings $F_k\colon H(\Omega)\to H(\Omega)$ defined by $F_k(f)=w_k f$ are homeomorphisms. Since also each h_k is entire and has no zeros outside Ω we conclude that for each $k=1,2,...,$ the set

$$\begin{aligned}F_k^{-1}(U_{h_k}(\Omega))&=\{f:w_k f\in U_{h_k}(\Omega)\}=\{f:h_k w_k f\in U(\Omega)\}\\ &=\{f\in H(\Omega):g_k f\in U(\Omega)\}\end{aligned} \tag{4.5}$$

is G_δ-dense in $H(\Omega)$. Now it suffices to take any $u\in U(\Omega)\cap\bigcap_{k=1}^{\infty} F_k^{-1}(U_{h_k}(\Omega))$ which is a nonempty countable intersection of G_δ-dense subsets of $H(\Omega)$ and set $u_k=g_k u$. Then clearly $u,u_1,u_2,...\in U(\Omega)$ and $g_k=\frac{u_k}{u}$ for $k=1,2,...,$ and so this completes the proof. ▮

5. SOME PROPERTIES OF UNIVERSAL TAYLOR SERIES

Let Ω be a simply connected domain, $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$. In the case where the admissible matrix A is the simple diagonal matrix $A = (a_{nk})_{n,k=0}^{\infty}$ with $a_{nk} = 0$ for $n \neq k$ and $a_{nn} = 1$, and the infinite set S is $S = \{0, 1, 2, \ldots\}$, the classes $U(\Omega, A, S, \zeta)$ and $U(\Omega, A, S)$ will be denoted by $U(\Omega, \zeta)$ and $U(\Omega)$, respectively (see Definitions 1.2 and 1.3). According to Theorem 3.4, these classes are G_δ-dense in $H(\Omega)$ (see also [19]). In this section we present some properties of these classes.

PROPOSITION 5.1. *Let Ω be a simply connected domain, $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, and $X \subset H(\Omega)$ that contains a G_δ-dense subset of $H(\Omega)$. Then, for every $f \in H(\Omega)$, there exist $u_1, u_2 \in X$ such that $f = u_1 - u_2$; in particular this holds for $X = U(\Omega)$.*

Proof. We consider the homeomorphism $G: H(\Omega) \to H(\Omega)$, where $G(g) = f + g$, for every $g \in H(\Omega)$. Then X and $G(X)$ both contain G_δ-dense subsets of $H(\Omega)$. Baire's theorem implies that $X \cap G(X) \neq \varnothing$. Let $u_1 \in X \cap G(X)$; then $u_1 \in X$ and $u_1 = f + u_2$ for some $u_2 \in X$. This gives the result. ∎

Remark. The above argument has been suggested by J.-P. Kahane (see also [18]). In the case where Ω is the open unit disk D and $X = U(D, 0)$ it has been proven in [16] that the Taylor coefficients of u_1, u_2 cannot be controlled by those of f. This is possible in a larger class, the class of Chui and Parnes and Luh [16].

Remark. Obviously $U(\Omega, \zeta)$ and $U(\Omega)$ are not closed under addition. We notice that in general those classes are also not closed under multiplication ([19]).

Remark. Let f be holomorphic in Ω without zeros. Then, as in the proof of Proposition 5.1, f can be written $f = \frac{u_1}{u_2}$ with $u_1, u_2 \in U(\Omega)$.

PROPOSITION 5.2. *Let Ω be a simply connected domain $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, and $\zeta \in \Omega$. Then for every $f \in U(\Omega, \zeta)$ the radius of convergence R of the Taylor development of f with center ζ is exactly $R = dist(\zeta, \Omega^c) < +\infty$.*

Proof. Obviously $R \geqslant dist(\zeta, \Omega^c)$. The circle $|z - \zeta| = dist(\zeta, \Omega^c)$ contains at least a point z_0 of $\partial\Omega$. Applying the definition of $U(\Omega, \zeta)$ to the compact set $K = \{z_0\}$ we see that the sequence $S_n(f, \zeta)(z_0)$, $n = 0, 1, 2, \ldots$, is dense in $\mathbb{C}$, and therefore divergent. This proves that $R = dist(\zeta, \Omega^c)$ and completes the proof. ∎

Remark. In the more general case, where $S=\{0, 1, 2, ...\}$ and the admissible matrix $A=(a_{nk}(z))_{n,k=0}^{\infty}$ is regular (that is, for every compact subset $\tilde{K}$ of V we have $\sum_{k=0}^{k(n)} |a_{nk}(z)| \leqslant \tilde{M} < +\infty$ for all $z \in \tilde{K}$ and $n=0, 1, 2, ...$), if $\sum_{n=0}^{\infty} a_n(f, \zeta)(z_0-\zeta)$ converges, it follows that the sequence $A_n(f, \zeta)(z_0)$, $n=0, 1, 2, ...,$ is bounded and therefore it is not dense. Hence we have the result of Proposition 5.2 in this more general case, also. The following counterexample shows that if A is not regular, this result may fail.

Counterexample. Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be any entire function with $a_n \neq 0$ for all $n \geqslant 0$, let $\{f_n(z)\}$ be an enumeration of all polynomials with coefficients in $Q+iQ$, and for $\Omega=D$ the unit disk centered at 0, $V=\mathbb{C}\backslash\{0\}$, $\zeta=0$, define the matrix $A=\{a_{nk}(z)\}$ by

$$a_{nk}(z)=\begin{cases} 1 & \text{if} \quad k=n-2 \\ -\dfrac{f_n(z)-f(z)}{a_n z^n} & \text{if} \quad k=n-1 \\ \dfrac{f_n(z)-f(z)}{a_n z^n} & \text{if} \quad k=n \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum_{k=0}^{n} a_{nk}(z)=1$ and it follows easily that A is admissible, whereas

$$\begin{aligned} A_n(f, 0)(z) &= \frac{f_n(z)-f(z)}{a_n z^n}\,[S_n(f, 0)(z)-S_{n-1}(f, 0)(z)]+S_{n-2}(f, 0)(z) \\ &= f_n(z)+S_{n-2}(f, 0)(z)-f(z); \end{aligned}$$

so since $S_{n-2}(f, 0)(z) \to f(z)$ uniformly on compact subsets of $\mathbb{C}$ and using Mergelyan's theorem, it is easy to see that $f \in U(D, A, 0)$, but certainly the radius of convergence of the Taylor development of f centered at 0 is strictly larger than 1 (it is infinite).

The following proposition relates to the Universal Trigonometric series of Menchoff [2, 17, 18].

Proposition 5.3. *Let Ω be a simply connected domain $\Omega \subset \mathbb{C}$, let $\Omega \neq C$, and let $\zeta \in \Omega$. We suppose that $(\partial\Omega)^c$ has a locally finite number of components. Let μ be any Radon measure on $\partial\Omega$ and let $f \in U(\Omega, \zeta)$. Then for every pair of μ-measurable functions h, g: $\partial\Omega \to [-\infty, +\infty]$, there exists a strictly increasing sequence $\lambda_n \in \{0, 1, 2, ...\}$, such that*

$$Re\, S_{\lambda_n}(f, \zeta)(z) \to h(z) \qquad \text{and} \qquad Im\, S_{\lambda_n}(f, \zeta)(z) \to g(z),$$

μ-almost everywhere as $n \to +\infty$.

Proof. The boundary $\partial\Omega$ of Ω is a closed subset of $\mathbb{C}$ with empty interior and its complement has locally finite number of components. Thus, according to Lemma 2.2, there exists an increasing sequence of compact sets K_n, $n=0, 1, 2, ...,$ with K_n^c connected and $K_n^0=\emptyset$, such that $\bigcup_{n=1}^{\infty} K_n=\partial\Omega$.

Let $h_n\colon \partial\Omega \to \mathbb{R}$ and $g_n\colon \partial\Omega \to \mathbb{R}$, $n=1, 2, ...,$ be two sequences of continuous functions, such that $h_n \to h$ and $g_n \to r$, μ-almost everywhere, as $n \to +\infty$. According to the definition of $U(\Omega, \zeta)$, for every $n=1, 2, ...,$ there exists a large natural number $\lambda_n (\lambda_n > \lambda_{n-1})$, such that $\sup_{z \in K_n} |S_{\lambda_n}(f, \zeta)(z) - (h_n(z)+ig_n(z))| < \frac{1}{n}$. The result now follows easily. ▮

Remark. The result of Proposition 5.3 easily extends to the class $U(\Omega, A, S, \zeta)$ when A is an admissible matrix in Ω.

PROPOSITION 5.4. *Let Ω be a simply connected domain, $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ and $\zeta \in \Omega$. If $f \in U(\Omega, \zeta)$, then f is not a rational function.*

Proof. We have seen that the radius of convergence of the Taylor development of f with center ζ is exactly $dist(\zeta, \Omega^c) = dist(\zeta, \partial\Omega) < +\infty$. We have also seen that for every $z_0 \in \partial\Omega$, the sequence $S_n(f, \zeta)(z_0)$, $n=0, 1, 2, ...$ is dense in $\mathbb{C}$; thus, we do not have $S_n(f, \zeta)(z_0) \to \infty$, as $n \to +\infty$.

On the other hand, if f is rational, then it has at least one pole on the set $\{z \in \mathbb{C} : |\zeta - z| = dist(\zeta, \partial\Omega)\}$. Pick such a pole z_0 of maximal multiplicity. Then $z_0 \in \partial\Omega$ and $S_n(f, \zeta)(z_0) \to \infty$, as $n \to +\infty$ [4, 16, 18]. This gives a contradiction and completes the proof. ▮

The following theorem relates $U(\Omega, \zeta)$ to (C, k) summability.

THEOREM 5.5. *Let Ω be a simply connected domain, $\Omega \subset C$, $\Omega \neq C$, and $\zeta \in \Omega$. Let $f \in U(\Omega, \zeta)$; then for every $z_0 \in \Omega^c$ the Taylor development $\sum_{n=0}^{\infty} a_n(f, \zeta)(z_0 - \zeta)^n$ of f at z_0 with center ζ is not (C, k) summable for any $k=1, 2, ...$. In particular this holds for every $z_0 \in \partial\Omega$.*

Proof. The radius of convergence of $\sum_{n=0}^{\infty} a_n(f, \zeta)(z_0 - \zeta)^n$ is $dist(\zeta, \partial\Omega) < +\infty$ (Proposition 5.2). For $z_0 \in \mathbb{C}$, such that $|z_0 - \zeta| > dist(\zeta, \partial\Omega)$, it is well known that $\sum_{n=0}^{\infty} a_n(f, \zeta)(z_0 - \zeta)^n$ is not (C, k) summable, for any $k=1, 2, ...$. Assume that $|z_0 - \zeta| = dist(\zeta, \partial\Omega)$ and so $z_0 \in \partial\Omega$.

Without loss of generality we may assume that $\zeta = 0$, $z_0 = 1 \in \partial\Omega$ and that $\{z\colon |z| < 1\} \subset \Omega$. We also assume that $\sum_{n=0}^{\infty} a_n(f, 0)\, z^n$ is (C, k) summable to $s \in \mathbb{C}$ at $z_0 = 1$ for some $k \in \{1, 2, ...,\}$ and we shall arrive at a contradiction.

We consider a compact set K, as in Lemma 2.8. We have $K \cap \Omega = \emptyset$ and K^c connected. Therefore, according to the definition of $U(\Omega, 0)$, there exists an infinite subset E of $\{1, 2, ...\}$ such that the sequence $S_n(z)$, $n \in E$ converges uniformly on K to the constant function $s+1$.

Let $\ell_1, ..., \ell_k$ be distinct natural numbers. According to Lemma 2.8, there exist an infinite subset $\mathscr{D} \subset E$ and complex numbers $z_{j,n} \in K$, $n \in \mathscr{D}$, $j = 1, ..., k$ such that $\lim_{n \to +\infty, n \in \mathscr{D}} [n(1 - z_{j,n})] = \zeta_j$ with $|\zeta_j| = \frac{1}{\ell_j}$, $j = 1, ..., k$.

Obviously $\zeta_j \neq 0$ for $j = 1, ..., k$ and the ζ_j, $j = 1, ..., k$, are distinct. Then according to Lemma 2.7 there exists $\lambda_{j,n} \in \mathbb{C}$, $j = 1, ..., k$, $n \in \mathscr{D}$, such that

$$\lim_{\substack{n \to +\infty \\ n \in \mathscr{D}}} \sum_{\ell=1}^{k} [z_{\ell,n}^{-1}(S_n(z_{\ell,n}) - s) - (S_n - s)] \, \lambda_{\ell,n} = 0$$

and

$$\lim_{\substack{n \to +\infty \\ n \in \mathscr{D}}} \lambda_{\ell,n} = \frac{1}{\zeta_j \prod_{r=1, r \neq j}^{k} (\zeta_j - \zeta_r)}, \qquad j = 1, ..., k.$$

One can easily see that $\lim_{n \to \infty, n \in \Delta} z_{j,n}^{-n} = e^{\zeta_j}$, $j = 1, ..., k$ and that $\lim_{n \to \infty, n \in \Delta} [S_n(z_{j,n}) - s] = \lim_{n \to \infty, n \in \Delta} (S_n - s) = 1$. Therefore we get

$$\sum_{\ell=1}^{k} \frac{e^{\zeta_\ell} - 1}{\zeta_\ell \prod_{r=1, r \neq \ell}^{k} (\zeta_\ell - \zeta_r)} = 0.$$

It follows that there exists a rational function R, depending only on $\zeta_2, ..., \zeta_k$, such that $e^{\zeta_1} = R(\zeta_1)$, for some ζ_1 satisfying $|\zeta_1| = \frac{1}{e_1}$. This holds for every natural number ℓ_1 different from $\ell_2, ..., \ell_k$. We fix $\ell_2, ..., \ell_k$ and $\zeta_2, ..., \zeta_k$ and we vary ℓ_1. By analytic continuation we conclude that the exponential function coincides with a rational one. This gives a contradiction and so the proof is complete. ∎

Remark. In the case $\Omega = \{z \in \mathbb{C} : |z| < 1\}$, Theorem 5.5 has been proven in [11, 16]. The method in [16] gives the result simultaneously for all points z_0, $|z_0| = 1$ and it is not possible to have the result only for some such points. The method in [11] enables one to distinguish some points on the circle of convergence, and it is this method that can be generalized.

A consequence of Theorem 5.5 is the following.

PROPOSITION 5.6. *Let Ω be a simply connected domain, $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$. Let $\zeta \in \Omega$ and $f \in U(\Omega, \zeta)$. Then f does not extend continuously on $\bar{\Omega}$. More precisely, for every $z_1 \in \partial\Omega$, for which there exists an open disk $D \subset \Omega$, such that $\bar{D} \cap \partial\Omega = \{z_1\}$, f does not extend continuously at z_1.*

Proof. If f extends continuously on $\bar{\Omega}$, then it is continuous on $\{z \in \mathbb{C} : |z - \zeta| \leqslant dist(\zeta, \partial\Omega)\}$ and holomorphic in its interior. Therefore, if

$z_0 \in \partial\Omega$, satisfies $|\zeta - z_0| = dist(\zeta, \partial\Omega)$, then the series $\sum_{n=0}^{\infty} a_n(f, \zeta)(z_0 - \zeta)^n$ is $(C, 1)$ summable. This contradicts Theorem 5.5.

If f is continuous on $\Omega \cup \{z_1\}$, then it is continuous on $\bar{D}$ and holomorphic in D. The previous argument applies also in this case and we have the result. ∎

The next proposition relates to a conjecture of J.-P. Kahane.

PROPOSITION 5.7. *Let Ω be a simply connected domain, $\Omega \subset \mathbb{C}$, $\Omega \neq C$. If $f \in U(\Omega)$, then f does not extend holomorphically to any disk D with $D \cap \Omega \neq \emptyset$ and $D \cap \Omega^c \neq \emptyset$.*

Proof. Suppose $\Omega \cap D \neq \emptyset$ and $D \cap \Omega^c \neq \emptyset$. Assume that f extends holomophically in $\Omega \cup D$. Let $z_0 \in (\partial\Omega) \cap D$ and W be an open disk with center z_0, such that $W \subset D \cup \Omega$. Then for $\zeta \in \Omega$ close to z_0, the radius of convergence of $\sum_{n=0}^{\infty} a_n(f, \zeta)(z - \zeta)^n$ is strictly greater than $|\zeta - z_0| \geqslant dist(\zeta, \partial\Omega)$. This contradicts Proposition 5.2. The proof is complete. ∎

The conjecture of Kahane states that if f belongs to the class $U(D, 0)$, where D is the open unit disk, then the unit circle is the natural boundary of f [10]. Section 8 below contains an affirmative answer to this conjecture.

Concerning the growth of the Taylor coefficients of universal Taylor series, the following result has been established in [16].

THEOREM 5.8. *Let $\sum_{n=0}^{\infty} a_n z^n$ be the Taylor development of any element of the class $U(D, 0)$, where D is the open unit disk. Let b_n, be any decreasing sequence such that $\sum_{n=1}^{\infty} \frac{b_n}{n}$ converges. Then $\lim_n \sup \frac{|a_n|}{e^{nb_n}} = +\infty$. In particular the sequence a_n, $n = 0, 1, 2, ...$ does not have polynomial growth and $U(D, 0)$ is disjoint from the Nevanlinna class.*

Versions of Theorem 5.8 have not yet been obtained when D is replaced by other simply connected domains or open sets.

6. DERIVATIVES AND ANTIDERIVATIVES

Let Ω be a simply connected domain, $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$. We denote by $D_0(\Omega)$ the set of $f \in H(\Omega)$, such that, for every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \emptyset$, with K^c connected and every function ϕ: $K \to \mathbb{C}$ continuous on K and holomorphic in K^o, there exists a sequence $\lambda_n \in \{0, 1, 2, ...\}$, $n = 1, 2, ...$ with the following properties:

(a) For every compact set $L \subset \Omega$ we have

$$\sup_{\zeta \in L} \sup_{z \in K} |S_{\lambda_n}(f, \zeta)(z) - \phi(z)| \to 0, \qquad \text{as} \quad n \to +\infty.$$

(b) For every compact set $\tilde{L}$, $\tilde{\tilde{L}} \subset \Omega$ we have

$$\sup_{\zeta \in \tilde{L}} \sup_{w \in \tilde{\tilde{L}}} |S_{\lambda_n}(f, \zeta)(w) - f(w)| \to 0, \text{ as } n \to +\infty.$$

According to Theorem 4.2 the class $D_0(\Omega)$ is G_δ-dense in $H(\Omega)$. Moreover, one can easily check that if P is any polynomial and $f \in D_0(\Omega)$, then $f + P$ belongs also to $D_0(\Omega)$. Thus Proposition 2.13 can be applied. For $k \in \{1, 2, ...\}$ we denote by $D_k(\Omega)$ the set of functions $f \in H(\Omega)$ such that the kth derivative $f^{(k)}$ belongs to $D_0(\Omega)$. It follows that $D_k(\Omega)$ is also G_δ-dense in $H(\Omega)$ (Proposition 2.13). We also fix a point $z_0 \in \Omega$. We define $f^{(-1)}(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta$. For $k \in \{-2, -3, ...\}$ we inductively define $f^{(k)}(z) = \int_{z_0}^{z} f^{(k+1)}(\zeta)\, d\zeta$. For $k \in \{-1, -2, ...\}$ we denote as $D_k(\Omega)$ the set of functions $f \in H(\Omega)$, such that $f^{(k)} \in D_0(\Omega)$. Then D_k, $k = -1, -2, ...$ is also G_δ-dense in $H(\Omega)$ (Proposition 2.13). Baire's theorem implies that $\bigcap_{k \in Z} D_k(\Omega)$ is G_δ-dense in $H(\Omega)$. Thus we have proved the following theorem (see also [15]).

THEOREM 6.1. *Let $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, be a simply connected domain. Then there exists a function $f \in H(\Omega)$ with the following property*:

For every $k \in Z$, every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \emptyset$ with K^c connected and every function ϕ: $K \to \mathbb{C}$ continuous on K and holomorphic in K^o, the exists a sequence $\lambda_n^k \in \{0, 1, 2, ...\}$, $n = 1, 2, ...$, such that:

(i) *For every compact set $L \subset \Omega$ we have*

$$\sup_{\zeta \in L} \sup_{z \in K} |S_{\lambda_n^k}(f^{(k)}, \zeta)(z) - \phi(z)| \to 0, \qquad \textit{as} \quad n \to +\infty.$$

(ii) *For every compact set $\tilde{L}$, $\tilde{\tilde{L}} \subset \Omega$ we have*

$$\sup_{\zeta \in \tilde{L}} \sup_{w \in \tilde{\tilde{L}}} |S_{\lambda_n^k}(f^{(k)}, \zeta)(w) - f^{(k)}(w)| \to 0, \qquad \textit{as} \quad n \to +\infty.$$

Furthermore, the set of functions $f \in H(\Omega)$ with this property is G_δ-dense in $H(\Omega)$.

7. NONCONNECTED SIMPLY CONNECTED OPEN SETS

So far the set $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$ was assumed to be a simply connected domain. We consider now the case where $\Omega \subset \mathbb{C}$ is a simply connected open set with several components Ω_j, $j \in J$, where the set J is finite or infinite denumerable. We consider the spaces $H(\Omega)$ and $H(\Omega_j)$, $j \in J$ with the topologies of uniform convergence on compacta. A function $f \in H(\Omega)$ can be identified with $(f_j)_{j \in J}$, where $f_j = f_{|\Omega_j}$. Thus $H(\Omega)$ as a set can be identified with the Cartesian product $\prod_{j \in J} H(\Omega_j)$. Furthermore, it is easily seen that the topology of $H(\Omega)$ coincides with the Cartesian topology of $\prod_{j \in J} H(\Omega_j)$. It is also well known that a denumerable cartesian product of G_δ-dense sets is G_δ-dense. Thus the set $\prod_{j \in J} U(\Omega_j)$ is G_δ-dense in $\prod_{j \in J} H(\Omega_j) \cong H(\Omega)$. Thus we have proved the following.

PROPOSITION 7.1. *Under the above assumptions and notations, there exists a function $f \in H(\Omega)$ with the following property:*

For every family of compact sets $K_j \subset \mathbb{C}$, $j \in J$, with $K_j \cap \Omega_j = \emptyset$ and K_j^c connected and every family of functions $\phi_j \colon K_j \to \mathbb{C}$, $j \in J$, continuous on K_j and holomorphic in K_j^o, there exists a family of sequences $\lambda_n^j \in \{0, 1, 2, ...\}$, $n = 1, 2, ...$, $j \in J$, such that for every compact set $L \subset \Omega$ and every $s \in J$ with $L \cap \Omega_s \neq \emptyset$ we have

$$\sup_{\zeta \in L \cap \Omega_s} \sup_{z \in K_s} |S_{\lambda_n^k}(f, \zeta)(z) - \phi_s(z)| \to 0, \qquad \text{as} \quad n \to +\infty.$$

Furthermore, the set of functions $f \in H(\Omega)$ with this property is G_δ-dense in $H(\Omega)$.

We shall strengthen the result of Proposition 7.1 by showing that the sequence λ_n^j, $n = 1, 2, ...$ can be chosen independently of $j \in J$.

DEFINITION 7.2. With the above notations and assumptions, a function $f \in H(\Omega)$ belongs to the class $U^*(\Omega)$, iff for every family of compact sets $K_j \subset \mathbb{C}$, $j \in J$, with $K_j \cap \Omega_j = \emptyset$ and K_j^c connected and every family of functions $\phi_j \colon K_j \to \mathbb{C}$, $j \in J$, continuous on K_j and holomorphic in K_j^o, there exists a sequence $\lambda_n \in \{0, 1, 2, ...\}$, $n = 1, 2, ...$, such that, for every compact set $L \subset \Omega$ and every $s \in J$ with $L \cap \Omega_s \neq \emptyset$ we have

$$\sup_{\zeta \in L \cap \Omega_s} \sup_{z \in K_s} |S_{\lambda_n}(f, \zeta)(z) - \phi_s(z)| \to 0, \qquad \text{as} \quad n \to +\infty.$$

THEOREM 7.3. *Under the above notations and assumptions the class $U^*(\Omega)$ is G_δ-dense in $H(\Omega)$.*

Sketch of the Proof. Let $L \subset \Omega$ be a compact set intersecting exactly the components $\Omega_{j_1}, ..., \Omega_{j_N}$. Let $K_{j_1}, ..., K_{j_N}$ be compact sets with connected complements such that $K_{j_1} \cap \Omega_{j_1} = \emptyset, ..., K_{j_N} \cap \Omega_{j_N} = \emptyset$.

Finally let $\phi_{j_1}: K_{j_1} \to \mathbb{C}, ..., \phi_{j_N}: K_{j_N} \to \mathbb{C}$ be continuous functions holomorphic in the interiors of their domains definition. We consider the set

$$E_n = \left\{ f \in H(\Omega) : \max_{i=1, ..., N} \sup_{\zeta \in L \cap \Omega_{j_i}} \sup_{z \in K_{j_i}} |S_n(f, \zeta)(z) - \phi_{j_i}(z)| < \frac{1}{s} \right\},$$

$n = 0, 1, 2, ...$ (where $s \in \{1, 2, ...\}$ is fixed). One easily can see that E_n is open in $H(\Omega)$. We shall also show that $\bigcup_{n=0}^{\infty} E_n$ is dense in $H(\Omega)$. Let $\omega \in H(\Omega)$, $\tilde{L} \subset \Omega$ compact with $\tilde{L}^c$ connected and $\varepsilon > 0$. We will find $n \in \{0, 1, 2, ...\}$ and $f \in E_n$, such that $\sup_{w \in \tilde{L}} |\omega(w) - f(w)| < \varepsilon$.

For each $i = 1, ..., N$, by Mergelyan's theorem we find a polynomial f_{j_i} such that $\sup_{w \in \tilde{L} \cap \Omega_{j_i}} |f_{j_i}(w) - \omega(w)| < \varepsilon$ and $\sup_{z \in K_{ji}} |f_{j_i}(z) - \phi_{j_i}(z)| < \frac{1}{s}$.

We choose n greater than the degrees of the polynomials f_{j_i}, $i = 1, ..., N$. We also consider the function $f \in H(\Omega)$ defined by $f|_{\Omega_{j_i}} = f_{j_i}$, $i = 1, ..., N$ and $f(z) = \omega(z)$ for $z \in \Omega \setminus \bigcup_{i=1}^{N} \Omega_{j_i}$. Then it is easy to check that this choice of n and f satisfies our requirement. Thus $\bigcup_{n=0}^{\infty} E_n$ is dense in $H(\Omega)$.

We fix the compact set L, as above, and we vary $s \in \{1, 2, ...\}$, ϕ_{j_i} to be polynomials with coefficients in $Q + iQ$, and the compact sets K_{j_i} according to Lemma 2.1. Then by denumerable intersection we obtain a G_δ-dense subset of $H(\Omega)$ depending on the compact set L. Finally varying L into an exhaustive sequence and taking the (denumerable) intersection we obtain the result. ∎

We also mention that a larger class $U(\Omega)$ can be defined according to Ddefinition 1.3 (with the only difference news that Ω is not connected). This class $U(\Omega)$ is residual and if Ω has a locally finite number of components, it is in fact G_δ-dense in $H(\Omega)$.

8. ON KAHANE'S CONJECTURE

This section contains an affirmative answer to the conjecture of Kahane mentioned above. The proof is based on a result in [5].

We recall Definition 1.2 in the introduction of the class $U(\Omega, \zeta)$. This definition can be considered even in the case where the domain Ω is not simply connected.

DEFINITION 8.1. Let $\sum_{\nu=0}^{\infty} a_\nu (z - \zeta)^\nu$ be a power series with radius of convergence $r \in (0, +\infty)$. We say that it has Ostrowski gaps (p_m, q_m) if (p_m) and (q_m) are sequences of natural numbers with

(a) $p_1 < q_1 \leqslant p_2 < q_2 \leqslant ...$ and $\lim_{m \to \infty} \frac{q_m}{p_m} = \infty$

(b) for $I = \bigcup_{m=1}^{\infty} \{p_m + 1, ..., q_m\}$ we have $\lim_{\nu \in I} |a_\nu|^{1/\nu} = 0$.

LEMMA 8.2. *Let Ω be a domain in $\mathbb{C}$ contained in some half-plane and let $\zeta_0 \in \Omega$. Let $u \in U(\Omega, \zeta_0)$ and $S_n(u, \zeta_0)(z)$, $n = 0, \ldots$ denote the partial sums of the Taylor development $u(z) = \sum_{v=0}^{\infty} a_v(u, \zeta_0) \cdot (z - \zeta_0)^v$. Let $K \subset \mathbb{C}$ be compact, such that $K \cap \Omega = \varnothing$ and K^c is connected. Also let $f: K \to \mathbb{C}$ be a continuous function on K which is holomorphic in K^o.*

Then there exists two sequences of natural numbers (n_k), (m_k) such that

(a) *The power series $\sum_{v=0}^{\infty} a_v(u, \zeta_0)(z - \zeta_0)^v$ has Ostrowski-gaps (n_k, m_k) and*

(b) $\sup_{z \in K} |S_{n_k}(u, \zeta_0)(z) - f(z)| \to 0$, *as* $k \to +\infty$

The proof of Lemma 8.2 is step1 in the proof of Theorem 4 in [5] even if K meets $\partial\Omega$.

THEOREM 8.3. *Let Ω be a domain in $\mathbb{C}$ contained in some half-plane and $\zeta_0, \zeta_1 \in \Omega$. Then we have*

(a) $U(\Omega, \zeta_0) \subset U(\Omega, \zeta_1)$ *and*

(b) $U(\Omega, \zeta_0) = \bigcap_{\zeta \in \Omega} U(\Omega, \zeta)$.

Proof. To prove (a) let $u \in U(\Omega, \zeta_0)$, let $K \subset \mathbb{C}$ be compact such that $K \cap \Omega = \varnothing$ and K^c is connected and let $f: K \to \mathbb{C}$ be continuous on K and holomorphic in K^o. According to Lemma 8.2, there exists two sequences of natural numbers (n_k), (m_k) such that

(i) The power series $\sum_{v=0}^{\infty} a_v(u, \zeta_0)(z - \zeta_0)^v$ has Ostrowski-gaps (n_k, m_k) and

(ii) $\sup_{z \in K} |S_{n_k}(u, \zeta_0)(z) - f(z)| \to 0$, as $k \to +\infty$.

According to Theorem 1 in [13] we have

$$\sup_{z \in K} |S_{n_k}(u, \zeta_0)(z) - S_{n_k}(u, \zeta_1)(z)| \to 0, \text{ as } k \to +\infty.$$

It follows that $\sup_{z \in K} |S_{n_k}(u, \zeta_1)(z) - f(z)| \to 0$, as $k \to +\infty$. Thus $u \in U(\Omega, \zeta_1)$ and the proof is complete. ∎

THEOREM 8.4. *Let Ω be a domain in $\mathbb{C}$ contained in some half-plane and $\zeta_0 \in \Omega$. If $u \in U(\Omega, \zeta_0)$, then u cannot be holomorphically extended in any domain G strictly containing Ω. In particular for $\Omega = D$ the open unit disk, Kahane's conjecture has an affirmative answer.*

Proof. Let $z_0 \in \partial\Omega$ and let W be an open disk centered at z_0 with radius $r > 0$. Assume that u is holomorphically extended in $\Omega \cup W$. We shall arrive at a contradiction.

Let $\zeta_1 \in \Omega$ such that $|z_0 - \zeta_1| < \frac{r}{2}$. It follows easily that the radius of convergence R of the Taylor development of u with center ζ_1 satisfies $R > \frac{r}{2} > |z_0 - \zeta_1| \geqslant dist(\zeta_1, \Omega^c)$.

According to Theorem 8.3 we have $u \in U(\Omega, \zeta_1)$. Proposition 5.2 implies that $R = dist(\zeta_1, \Omega^c)$. This gives a contradiction and the proof is complete. ∎

COROLLARY 8.5. *Let Ω be a domain in $\mathbb{C}$ contained in some half-plane and $\zeta \in \Omega$. Then the following are equivalent:*

(a) *Ω is simply connected*

(b) *$U(\Omega, \zeta)$ is G_δ-dense in $H(\Omega)$ with the topology of uniform convergence on compacta*

(c) *$U(\Omega, \zeta) \neq \emptyset$*

In particular, if Ω is an open annulus we have $U(\Omega, \zeta) = \emptyset$.

Proof. The implication (a) ⇒ (b) follows from Theorem 3.4. The implication (b) ⇒ (c) is obvious. Assume (c) to prove (a).

Since $U(\Omega, \zeta) \neq \emptyset$, there exists $u \in U(\Omega, \zeta)$. According to Theorem 8.4 u is holomorphic exactly on Ω.

Lemma 8.2 implies that the Taylor development of u with center ζ has Ostrowski-gaps. By the special case of Ostrowski's Theorem (Theorem 16.7.2 of [6]), this power series has a single–valued analytic extension onto a simply connected domain $\Omega^* \supset \Omega$ such that it is holomorphic exactly on Ω^*. Thus $\Omega = \Omega^*$ is simply connected. This completes the proof. ∎

The last argument in the proof of Corollary 8.5 has been given by [5]. Finally we close with the following.

COROLLARY 8.6. *Let Ω be a simply connected domain which is contained in a half-plane. Let $z_0 \in \partial\Omega$, such that there exists an open disk $W \subset \Omega$ satisfying $\overline{W} \cap \partial\Omega = \{z_0\}$. Let also $\zeta_0 \in \Omega$ and $u \in U(\Omega, \zeta_0)$. Then $\lim_{z \to z_0, z \in \Omega} u(z)$ does not exist in $\mathbb{C}$. In the particular case where $\Omega = D$ is the open unit disk, the previous holds for every $z_0 \in \partial D$.*

Proof. Suppose $\lim_{z \to z_0, z \in \Omega} u(z) \in \mathbb{C}$ to arrive at a contradiction. Let ζ_1 be the center of the disk W. Then u is continuous on $\overline{W}$ and holomorphic in W. Therefore the Taylor development of u with center ζ_1 is $(C, 1)$ summable at z_0. However, since $u \in U(\Omega, \zeta_1)$ according to Theorem 8.3, it follows that the Taylor development of u with center ζ_1 is not $(C, 1)$ summable at $z_0 \in \partial\Omega$ (Theorem 5.5). This gives a contradiction.

In the case $\Omega = D$, the previous holds for every $z_0 \in \partial D$, because the disk W centered at $\frac{z_0}{2}$ with radius $\frac{1}{2}$ has the desired properties. This completes the proof. ∎

Corollary 8.6 gives a stronger statement than Kahane's conjecture.

9. ON THE CLASSES $U(\Omega, \zeta)$ AND $U(\Omega)$

In this section we will show that for any domain Ω contained in some half-plane the classes $U(\Omega, \zeta)$ and $U(\Omega)$ defined in Def. 1.2 and 1.3 actually coincide. The proof is based on the results in [5]. We have the following:

THEOREM 9.1. *Let $\Omega \subseteq \mathbb{C}$ be a domain contained in some half-plane and let $\zeta_0 \in \Omega$. Then $U(\Omega, \zeta_0) = U(\Omega)$. Actually the following holds: Let $f \in U(\Omega, \zeta_0)$. Then for every compact set $K \subseteq \Omega^c$ with K^c connected and every function h continuous on K and holomorphic in K^o there exists a sequence (p_k) of natural numbers such that for every compact set $L \subset \Omega$:*

(i) *$S_{p_k}(f, \zeta)(z) \to h(z)$ uniformly for $(\zeta, z) \in L \times K$.*

(ii) *$S_{p_k}(f, \zeta)(z) \to f(z)$ uniformly for $(\zeta, z) \in L \times L$.*

Proof. According to the proof of Theorem 4 in [5] (step 1) there exist sequences (p_k), (q_k) such that $f(z) = \sum_{\nu=0}^{\infty} a_\nu (z - \zeta_0)^\nu$ has Ostrowski gaps (in the sense of Definition 8.1) (p_k, q_k) and $S_{p_k}(f, \zeta_0)(z) \to h(z)$ uniformly on K (the proof works even if K meets $\partial\Omega$). We will show in Lemma 9.2 below that this implies that $S_{p_k}(f, \zeta)(z) - S_{p_k}(f, \zeta_0)(z)$ converges to zero uniformly for $(\zeta, z) \in L \times (K \cup L)$. From this the conclusion follows easily [13].

Hence the proof will be complete once we have proven the following (see Theorem 1 in [13]).

LEMMA 9.2. *Let $f(z) = \sum_{\nu=0}^{\infty} a_\nu^{(\zeta_0)} (z - \zeta_0)^\nu$ be the Taylor development of a holomorphic function on an open set Ω around the point $\zeta_0 \in \Omega$. Suppose that this series has Ostrowski gaps (p_k, q_k) in the sense of Definition 8.1. Then the difference $S_{p_k}(f, \zeta)(z) - S_{p_k}(f, \zeta_0)(z)$ converges to zero (as $k \to \infty$) uniformly on compact sets of $\Omega \times \mathbb{C}$ ($\zeta \in \Omega$, $z \in \mathbb{C}$).*

Proof. The proof will be that of Theorem 1 in [13] with minor modifications to obtain uniformity with respect to the center ζ. By Definition 8.1 $f(z) = \tilde{f}(z) + \omega(z)$, where $\omega(z)$ is entire and $a_\nu(\tilde{f}, \zeta) = 0$ for $p_k < \nu < q_k$ so we may assume without loss of the generality that $a_\nu^{(\zeta_0)} = 0$

for $p_k < v < q_k$. Let $\zeta_0 \in L \subseteq B^0 \subseteq B \subseteq \Omega$, L, B be compact and $2\eta = dist(L, \partial B) > 0$. Then as in the proof of Theorem 1 in [13] there exist constants $C > 0$, $0 < \delta < 1$ and $k_0 \geqslant 1$ that depend on f and B (and not on L) such that $\max_{|z-\zeta| \leqslant \eta} |f(z) - S_{p_k}(f, \zeta)(z)| < C^{p_k}\delta^{q_k}$ for every $k \geqslant k_0$ and every $\zeta \in L$, and also

$$\max_{|z-\zeta| \leqslant 4R} (|S_{p_k}(f, \zeta)(z)| + |S_{p_k}(f, \zeta_0)(z)|) \leqslant C_1 \tilde{R}^{p_k+1}$$

for every $\zeta \in L$ and $R \geqslant 1 + 2\eta$ where $\tilde{R} = \frac{1}{2\eta}(R + \max_{\zeta \in L} |\zeta - \zeta_0|)$. Choosing now R sufficiently large so that $|z - \zeta| \leqslant 2R$ for all $\zeta \in L$ and z with $|z| \leqslant R$ the proof can be completed as in [13] (and uniformly in $\zeta \in L$) by using the three-circles theorem. ∎

ACKNOWLEDGEMENTS

We express our gratitude to J.–P. Kahane, G. Koumoullis, W. Luh, S. Ruscheweyh, and M. Zinsmeister.

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